Exercise Notes: For Exercise 5, Proposition 3.3.16 allows one to construct the required subsequence as follows: For $k = 1$ let n_1 be such that $s_{n_1} > 1$; for $k = 2$ let $n_2 > n_1$ be such that $s_{n_2} > 2$; etc. For 5(b), Proposition 3.3.18 allows one to construct the desired subsequence as well. For Exercise 12, observe that for any $N \in \mathbb{N}$ the set $\{n \in \mathbb{N} : \sigma(n) \leq N\}$ is finite. This exercise implies that if a sequence $\langle s_n \rangle$ converges to c, then then any sequence obtained by reordering the terms in a subsequence of $\langle s_n \rangle$ also converges to c.

3.4 Monotone Sequences

An *increasing sequence* is one in which each term is less than or equal to the term after it. An *decreasing sequence* is one in which each term is greater than or equal to the term after it.

Definition 3.4.1. A sequence $\langle s_n \rangle$ is an **increasing sequence** if for all $n \in \mathbb{N}$ we have that $s_n \leq s_{n+1}$. A sequence $\langle s_n \rangle$ is a **decreasing sequence** if for all $n \in \mathbb{N}$ we have that $s_n \geq s_{n+1}$. A sequence is **monotone** if it is either increasing or decreasing.

Our first lemma shows that if a sequence is increasing and bounded above, then the supremum of the sequence is its limit.

Lemma 3.4.2. If $\langle s_n \rangle$ is increasing, bounded above and $\beta = \sup\{s_n : n \in \mathbb{N}\}$, then $\lim\limits_{n\to\infty} s_n = \beta$.

Proof. Suppose that $\langle s_n \rangle$ is an increasing and bounded sequence. Thus, whenever $m < n$ we have that $s_m \leq s_n$. Let $S = \{s_n : n \in \mathbb{N}\}$. Since S is bounded above, let $\beta = \sup(S)$. Thus, $s_n \leq \beta$ for all $n \in \mathbb{N}$. We will show that $\lim\limits_{n\to\infty} s_n = \beta$. To do this, let $\varepsilon > 0$. Because $\beta - \varepsilon < \beta$ and $\beta = \sup(S)$, it follows that there is an $N \in \mathbb{N}$ such that $\beta - \varepsilon < s_N$. Thus, (#) $\beta - s_N < \varepsilon$. Let $n > N$. Since $n > N$, we have that $s_N \leq s_n$. Thus, $-s_n \leq -s_N$ and so, ($\star$) $\beta - s_n \leq \beta - s_N$. We now prove that $|s_n - \beta| < \varepsilon$ as follows:

$$\begin{aligned} |s_n - \beta| &= \beta - s_n && \text{because } s_n \leq \beta \\ &\leq \beta - s_N < \varepsilon && \text{by } (\star) \text{ and } (\#). \end{aligned}$$

This completes the proof of the theorem. □

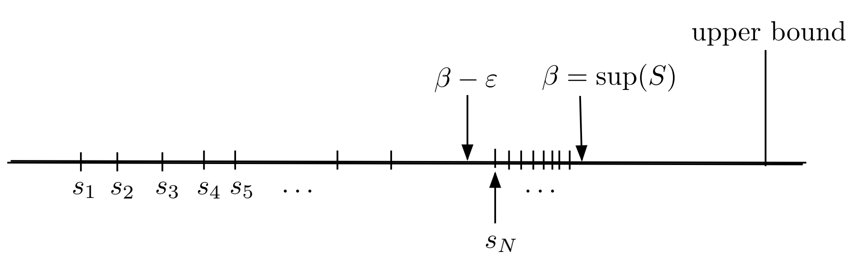

Figure 3.4: Representation of the proof of Lemma 3.4.2

Our next lemma shows that if a sequence is decreasing and bounded below, then its infimum is the limit.

Lemma 3.4.3. If $\langle s_n \rangle$ is decreasing, bounded below and $\alpha = \inf\{s_n : n \in \mathbb{N}\}$, then $\lim\limits_{n\to\infty} s_n = \alpha$.

Proof. This is an exercise. □

Theorem 3.4.4 (Monotone Convergence Theorem)**.** Suppose $\langle s_n \rangle$ is a monotone sequence. Then $\langle s_n \rangle$ is convergent if and only if $\langle s_n \rangle$ is bounded.

Proof. Let $\langle s_n \rangle$ be a monotone sequence. If $\langle s_n \rangle$ is convergent, then Theorem 3.1.25 implies that the sequence is bounded. If $\langle s_n \rangle$ is bounded, then either Lemma 3.4.2 or Lemma 3.4.3 implies $\langle s_n \rangle$ is convergent. □

Corollary 3.4.5. If $\langle s_n \rangle$ is increasing and $\lim_{n\to\infty} s_n = s$, then $s_n \le s$ for all $n \ge 1$.

Proof. If $\langle s_n \rangle$ is increasing and $\lim_{n\to\infty} s_n = s$, then $s = \sup\{s_n : n \in \mathbb{N}\}$ and $s_n \le s$ for all $n \ge 1$, by Lemma 3.4.2. □

Corollary 3.4.6. If $\langle s_n \rangle$ is decreasing and $\lim_{n\to\infty} s_n = s$, then $s_n \ge s$ for all $n \ge 1$.

Problem 3.4.7. Inductively define the sequence $\langle s_n \rangle$ by $s_1 = 2$ and $(\star)$ $s_{n+1} = \sqrt{6 + s_n}$ for all $n \ge 1$. Prove by induction that the sequence is monotone and bounded. Using the Monotone Convergence Theorem show that the sequence $\langle s_n \rangle$ converges, and then find its limit.

Solution. First we use mathematical induction to prove the following proposition.

Proposition 3.4.7.1. For every natural number $n \ge 1$, $0 < s_n \le s_{n+1} \le 10$.

Proof. We use mathematical induction.

Base step: For $n = 1$, we have $s_1 = 2$ and $s_2 = \sqrt{6 + s_1} = \sqrt{8} = 2\sqrt{2}$. Thus, $0 < s_1 \le s_2 \le 10$.

Inductive step: Let $n \ge 1$ and assume the induction hypothesis that

$$0 < s_n \le s_{n+1} \le 10. \tag{IH}$$

We prove that $0 < s_{n+1} \le s_{n+2} \le 10$. To prove that $0 < s_{n+1} \le s_{n+2} \le 10$, note that $(\star)$ implies $s_{n+1} = \sqrt{6 + s_n}$ and $s_{n+2} = \sqrt{6 + s_{n+1}}$. Thus,

$$\begin{aligned}
0 < s_n \le s_{n+1} &\le 10 && \text{by (IH)}\\
0 < 6 + s_n \le 6 + s_{n+1} &\le 6 + 10 && \text{by prop. of inequality}\\
0 < \sqrt{6 + s_n} \le \sqrt{6 + s_{n+1}} &\le \sqrt{16} && \text{by prop. of inequality}\\
0 < s_{n+1} \le s_{n+2} &\le 4 && \text{by } (\star).
\end{aligned}$$

Hence, $0 < s_{n+1} \le s_{n+2} \le 10$. This completes the proof. □

We can now conclude that the sequence $\langle s_n \rangle$ is monotone and bounded. Thus, by the Monotone Convergence Theorem 3.4.4 we know that $\langle s_n \rangle$ converges. Let s satisfy $\lim_{n\to\infty} s_n = s$. We are asked to find the numeric value of s. Since $s_{n+1} = \sqrt{6 + s_n}$, we conclude that $(\dagger)$ $s_{n+1}^2 = 6 + s_n$. We note that since $\lim_{n\to\infty} s_n = s$, we also have that $\lim_{n\to\infty} s_{n+1} = s$ by Theorem 3.3.5, and thus $\lim_{n\to\infty} s_{n+1}^2 = s^2$ by Theorem 3.2.3. Thus,

$$\begin{aligned}
s^2 = \lim_{n\to\infty} s_{n+1}^2 &= \lim_{n\to\infty} (6 + s_n) && \text{by } (\dagger)\\
&= 6 + \lim_{n\to\infty} s_n && \text{by Exercise 10 on page 55}\\
&= 6 + s && \text{because } \lim_{n\to\infty} s_n = s.
\end{aligned}$$

Therefore, we have the equation $s^2 = 6 + s$ and thus, $s^2 - s - 6 = 0$. The roots of this equation are $s = -2, 3$. Since the limit of a sequence with positive terms cannot be negative (see Theorem 3.2.12), we must have $s = 3$. Therefore, $\lim_{n\to\infty} s_n = 3$.

Theorem 3.4.8. The sequence $\left\langle (1 + \frac{1}{n})^n \right\rangle$ converges to a real number between 2 and 3.

Proof. We shall show that the sequence $\langle(1+\frac{1}{n})^n\rangle$ is increasing and bounded above by 3. By the Binomial Theorem 1.4.11, for all natural numbers $m \geq 3$, we have that

$$\left(1+\frac{1}{m}\right)^m = \sum_{k=0}^{m}\binom{m}{k}\frac{1}{m^k} = 1+1+\sum_{k=2}^{m}\binom{m}{k}\frac{1}{m^k} = 1+1+\left(\sum_{k=2}^{m-1}\binom{m}{k}\frac{1}{m^k}\right)+\frac{1}{m^m}. \tag{3.16}$$

One can check that $2=(1+\frac{1}{1})^1<(1+\frac{1}{2})^2<(1+\frac{1}{3})^3$. We will show that $(1+\frac{1}{n})^n<(1+\frac{1}{n+1})^{n+1}$ for all $n \geq 3$. Let $n \geq 3$. Then

$$\begin{aligned}
\left(1+\frac{1}{n}\right)^n &= 1+1+\sum_{k=2}^{n}\binom{n}{k}\frac{1}{n^k} && \text{by (3.16)}\\
&= 1+1+\sum_{k=2}^{n}\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) && \text{by Exercise 6}\\
&< 1+1+\sum_{k=2}^{n}\frac{1}{k!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right) && \text{by property of inequality}\\
&= 1+1+\sum_{k=2}^{n}\binom{n+1}{k}\frac{1}{(n+1)^k} && \text{by Exercise 6}\\
&< 1+1+\left(\sum_{k=2}^{n}\binom{n+1}{k}\frac{1}{(n+1)^k}\right)+\frac{1}{(n+1)^{n+1}} && \text{as } \frac{1}{(n+1)^{n+1}}>0\\
&= 1+1+\sum_{k=2}^{n+1}\binom{n+1}{k}\frac{1}{(n+1)^k} && \text{by (3.16)}\\
&= \left(1+\frac{1}{n+1}\right)^{n+1} && \text{by (3.16).}
\end{aligned}$$

Therefore, $(1+\frac{1}{n})^n<(1+\frac{1}{n+1})^{n+1}$ and the sequence is increasing. Moreover, (3.16) and Exercise 6 imply, for $n \geq 2$, that (see Theorem 1.4.5 and 1.4.6)

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k} = 2+\sum_{k=2}^{n}\binom{n}{k}\frac{1}{n^k} \leq 2+\sum_{k=2}^{n}\frac{1}{2^{k-1}} < 3.$$

Theorem 3.4.4 and Lemma 3.4.2 imply that $\langle(1+\frac{1}{n})^n\rangle$ converges to a limit between 2 and 3. □

3.4.1 The Monotone Subsequence Theorem

In this section we shall prove that every sequence of real numbers has a monotone subsequence. First we define what it means for a term in a sequence to be a "peak."

Definition 3.4.9. Let $\langle s_n\rangle$ be a sequence of real numbers. Let s_m be the m-th term of this sequence. We say that s_m is a *peak* if $s_m > s_n$ for all $n > m$ (see Figure 3.5).

Remark. Let $\langle s_n\rangle$ be a sequence of real numbers. Let s_m be the m-th term of this sequence. Then s_m is not a peak means that for some $n > m$ we have that $s_m \leq s_n$.

Example 3.4.10. Consider the sequence $\langle s_n\rangle = \left\langle\frac{(-1)^n}{n}\right\rangle$. So, $\langle s_n\rangle = \langle -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \dots\rangle$. Thus, $s_4 = \frac{1}{4}$ is a peak and $s_5 = -\frac{1}{5}$ is not a peak because $s_5 = -\frac{1}{5} < \frac{1}{6} = s_6$. Let $P = \{s_m : s_m \text{ is a peak}\}$. Then $P = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots\}$ and P is infinite.

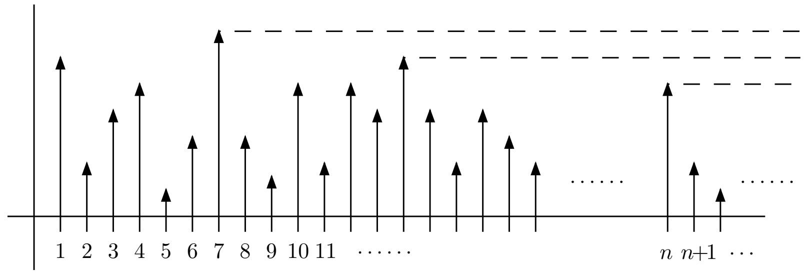

Figure 3.5: s_7, s_{14} and s_n are peaks; s_4 and s_8 are not a peaks.

Example 3.4.11. Consider the sequence $\langle s_n \rangle = \left\langle 2, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots \right\rangle$. Thus, $s_1 = 2$ is a peak and $s_3 = \frac{1}{2}$ is not a peak since $s_3 = \frac{1}{2} < \frac{3}{4} = s_5$. Let $P = \{s_m \mid s_m \text{ is a peak}\}$. Then $P = \{2, 1\}$ and P is finite.

Theorem 3.4.12 (Monotone Subsequence Theorem)**.** Every sequence of real numbers has a monotone subsequence.

Proof. Let $\langle s_n \rangle$ be a sequence of real numbers. Let $P = \{s_m \mid s_m \text{ is a peak}\}$. There are two cases to consider: Either P is infinite or P is finite.

Case 1: P is infinite. Since P is infinite, we can construct a subsequence of $\langle s_n \rangle$, consisting of peaks, as follows: Let m_1 be the first index such that s_{m_1} is a peak; that is, let s_{m_1} be the first peak. Let m_2 be the smallest natural number larger than m_1 such that s_{m_2} is a peak; that is, let s_{m_2} be the second peak. Since P is infinite, we can continue in this manner obtaining $m_1 < m_2 < m_3 < \cdots < m_k < \cdots$ where the subsequence $\langle s_{m_k} \rangle$ is such that each s_{m_k} is a peak. Since each s_{m_k} is a peak, we conclude that $s_{m_1} > s_{m_2} > s_{m_3} > \cdots > s_{m_k} > \cdots$ and thus, we have constructed a monotone subsequence.

Case 2: P is finite. Since P is finite, let $P = \{s_{m_1}, s_{m_2}, \dots, s_{m_r}\}$ be a finite listing of all the peaks. We can construct a subsequence of $\langle s_n \rangle$, consisting of terms that are not peaks, as follows: Let n_1 be larger than all of natural numbers in $\{m_1, m_2, \dots, m_r\}$. It follows that s_{n_1} is not a peak and that s_k is not a peak for all $k \geq n_1$. Since s_{n_1} is not a peak, there is a natural number $n_2 > n_1$where $s_{n_1} \leq s_{n_2}$. Now, since s_{n_2} is not a peak, there is a natural number $n_3 > n_2$ where $s_{n_2} \leq s_{n_3}$. We can continue in this manner obtaining $n_1 < n_2 < n_3 < \cdots < n_k < \cdots$ where the subsequence $\langle s_{n_k} \rangle$ is such that each s_{n_k} is not a peak and $s_{n_1} \leq s_{n_2} \leq s_{n_3} \leq \cdots \leq s_{n_k} \leq \cdots$ and thus, we have a monotone subsequence. ☐

Exercises 3.4

1. Give an example of a sequence that converges and is not monotone.
2. Show that the sequence $\left\langle n + \frac{(-1)^n}{n} \right\rangle$ is a increasing sequence. Prove that this sequence does not converge.
3. Prove Lemma 3.4.3.
4. Prove Corollary 3.4.6.
5. Inductively define the sequence $\langle s_n \rangle$ by $s_1 = 1$ and $s_{n+1} = \frac{1}{4}(2s_n + 3)$ for all $n \geq 1$. Prove by induction that the sequence is monotone and bounded. Using the Monotone Convergence Theorem show that the sequence $\langle s_n \rangle$ converges, and then find its limit.
6. Let $m \in \mathbb{N}$ and $k \in \mathbb{N}$ be such that $m \geq k \geq 2$. Using algebra, show that
$$\binom{m}{k} \frac{1}{m^k} = \frac{1}{k!} \left(1 - \frac{1}{m}\right) \left(1 - \frac{2}{m}\right) \cdots \left(1 - \frac{k-1}{m}\right).$$

Now conclude that $\binom{m}{k}\frac{1}{m^k} < \frac{1}{k!} \le \frac{1}{2^{k-1}}$.

7. Let $A \subseteq \mathbb{R}$ be nonempty and bounded. Let $\beta = \sup(A)$. Thus, for each $n \in \mathbb{N}$ there is an $b_n \in A$ such that $\beta - \frac{1}{n} < b_n$, and by Theorem 3.4.12 the sequence $\langle b_n \rangle$ has a monotone subsequence $\langle b_{n_k} \rangle$.

(a) Show that $\lim_{n\to\infty} b_n = \beta$.

(b) Prove that $\lim_{k\to\infty} b_{n_k} = \beta$.

(c) Suppose that $\beta \notin A$. Prove that $\langle b_{n_k} \rangle$ must be an increasing sequence.

8. Let $A \subseteq \mathbb{R}$ be nonempty and bounded. Let $\alpha = \inf(A)$. Thus, for each $n \in \mathbb{N}$ there is an $a_n \in A$ such that $a_n < \alpha + \frac{1}{n}$, and by Theorem 3.4.12 the sequence $\langle a_n \rangle$ has a monotone subsequence $\langle a_{n_k} \rangle$.

(a) Show that $\lim_{n\to\infty} a_n = \alpha$.

(b) Prove that $\lim_{k\to\infty} a_{n_k} = \alpha$.

(c) Suppose that $\alpha \notin A$. Prove that $\langle a_{n_k} \rangle$ must be a decreasing sequence.

3.5 Bolzano–Weierstrass Theorems

Bolzano–Weierstrass Theorem for sequences is a fundamental result about convergence which states that each bounded sequence in $\mathbb{R}$ has a convergent subsequence. This theorem is named after the mathematicians Bernard Bolzano and Karl Weierstrass. It was first proved by Bolzano, but his proof was lost. It was re-proven by Weierstrass and became an important centerpiece of analysis.

Theorem 3.5.1 (Bolzano–Weierstrass Theorem for sequences)**.** If the sequence $\langle s_n \rangle$ is bounded, then $\langle s_n \rangle$ has a convergent subsequence.

Proof. We are assuming that the sequence $\langle s_n \rangle$ is bounded. By the Monotone Subsequence Theorem 3.4.12, there is a monotone subsequence $\langle s_{n_k} \rangle$. Since $\langle s_n \rangle$ is bounded, it follows that $\langle s_{n_k} \rangle$ is bounded. Because $\langle s_{n_k} \rangle$ is a bounded monotone sequence, the Monotone Convergence Theorem 3.4.4 implies that $\langle s_{n_k} \rangle$ is a convergent subsequence. □

Definition 3.5.2. Let S be a subset of $\mathbb{R}$.

1. A point $x \in \mathbb{R}$ is an **accumulation point** of S if every neighborhood of x contains an infinite number of points from S. That is, if U is any neighborhood of x, then $S \cap U$ is infinite.
2. A point $x \in \mathbb{R}$ is an **isolated point of** S if $x \in S$ and x is not an accumulation point of S.

A point $x \in \mathbb{R}$ is an *accumulation point* of a set S if there are always an infinite number of points from the set S that are "very close" to x; that is, in every neighborhood of x. Thus, if $x \in I$ and I is an interval, then x is an an accumulation point of I.

A point x is an an *isolated point* of S if there is a neighborhood of x in which there are no other points from the set S (x is all alone; that is, x is the only point from S living in this neighborhood).

Remark 3.5.3. An accumulation point of S may be in the set S or may not be in S. On the other hand, an an isolated point must be in S.

Problem 3.5.4. For each of the following subsets S of $\mathbb{R}$ find some accumulation points (if any) and find some isolated points.

1. $S = [0, 3)$.