Chapter 3

Sequences

Sequences are fundamental in real analysis and, while you may already be familiar with sequences, it is useful to have a formal definition. We shall define a sequence to be just a function from the set of natural numbers into the set of real numbers $\mathbb{R}$.

Definition 3.0.1. A **sequence** is a function $s\colon \mathbb{N} \to \mathbb{R}$. We shall denote the value $s(n)$ by s_n, where s_n is called the **n-th term** of the sequence. We will write s as $\langle s_n \rangle$, as $\langle s_1, s_2, s_3, \dots \rangle$, or as $\langle s_n \rangle_{n=1}^{\infty}$ when we want to emphasize that the index variable n begins with 1.

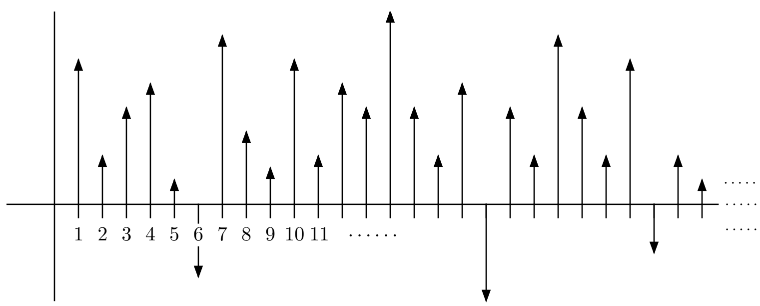

Figure 3.1: Functional representation of a sequence: $s(4) = s_4 > 0$ and $s(6) = s_6 < 0$

Consider the sequence $\langle s_n \rangle$ where $s_n = \frac{1}{n}$. Then we can write $\langle s_n \rangle$ as $\left\langle \frac{1}{n} \right\rangle$ or as $\left\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\rangle$. A constant sequence is denoted by $\langle a \rangle$ or $\langle a, a, a, a, \dots \rangle$, where a is a fixed real number.

One can also have sequences of the form $\langle s_n \rangle_{n=k}^{\infty} = \langle s_k, s_{k+1}, \dots \rangle$ where $k > 1$. However, one can easily re-express this sequence as one starting at 1. Define $\langle t_n \rangle_{n=1}^{\infty}$ by $t_n = s_{n+k-1}$ for all $n \geq 1$. Then $\langle t_n \rangle_{n=1}^{\infty} = \langle s_k, s_{k+1}, \dots \rangle$. For example, $\langle \frac{1}{n-1} \rangle_{n=2}^{\infty} = \langle \frac{1}{n} \rangle_{n=1}^{\infty}$.

3.1 Convergence

The limit of a sequence is one of the oldest and among the most important concepts in mathematical analysis. A sequence converges to the limit ℓ if the terms of the sequence get closer and closer to the real number ℓ. We now give a precise definition of this concept.

Definition 3.1.1. A sequence $\langle s_n \rangle$ is said to **converge** to the real number ℓ provided that for all $\varepsilon > 0$ there exists a natural number N such that for all $n \in \mathbb{N}$, if $n > N$ then $|s_n - \ell| < \varepsilon$.

Figure 3.2 illustrates Definition 3.1.1. If a sequence $\langle s_n \rangle$ converges to ℓ, then ℓ is called the **limit** of the sequence $\langle s_n \rangle$ and we write $\lim\limits_{n\to\infty} s_n = \ell$. If a sequence $\langle s_n \rangle$ does not converge, then we shall say that $\langle s_n \rangle$ **diverges**. The logical form of Definition 3.1.1 can be expressed as

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \to |s_n - \ell| < \varepsilon) \tag{3.1}$$

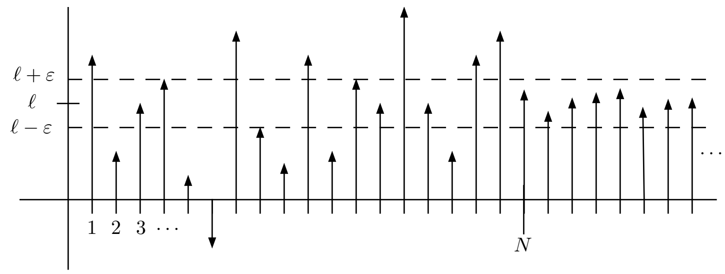

Figure 3.2: For all $n > N$ we have $|s_n - \ell| < \varepsilon$.

and it is this logical form that motivates the following proof strategy.

Proof Strategy 3.1.2. To prove that $\lim_{n\to\infty} s_n = \ell$, we will use the proof diagram

Let $\varepsilon > 0$ be an arbitrary real number.
  Let $N =$ (the natural number you found).
    Let $n > N$ be an arbitrary natural number.
      Prove $|s_n - \ell| < \varepsilon$.

To apply proof strategy 3.1.2 on a specific sequence, first let $\varepsilon > 0$. We must find a natural number N such that when $n > N$, we can prove that $|s_n - \ell| < \varepsilon$. To find the desired N, we will first attempt the following:

> *Using algebra and properties of inequality on the expression $|s_n - \ell|$, 'extract out' a larger value that resembles $\frac{1}{n}$.*

We shall then use this larger value to find N so that when $n > N$ we will have that $|s_n - \ell| < \varepsilon$. We will illustrate this idea in our proof analysis of the next four theorems. Before we discuss these theorems, we identify three properties of inequality that are very useful when proving theorems about convergence.

Quotient Principles of Inequality 3.1.3. Let a, b, c, d be positive real numbers. Then:

(1) Given the ratio $\frac{a}{b}$, you can conclude that $\frac{a}{b} < \frac{c}{b}$, if $a < c$.
(Replacing a numerator with a larger value yields a larger ratio.)

(2) Given the ratio $\frac{a}{b}$, you can conclude that $\frac{a}{b} < \frac{a}{d}$, if $d < b$.
(Replacing a denominator with a smaller value yields a larger ratio.)

(3) Given the ratio $\frac{a}{b}$, you can conclude that $\frac{a}{b} \le \frac{c}{d}$, if $a \le c$ and $d \le b$.
(Replacing a numerator with a larger value and denominator with a smaller value yields a larger ratio.)

Example 3.1.4. The property in the above 3.1.3(2) implies the following assertions:

1. $\frac{1}{n} < \frac{1}{N}$ when $n > N > 0$.
2. $\frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}}$ when $n > N > 0$, by Theorem 1.1.16 on page 11.
3. $\frac{1}{2^n} < \frac{1}{n}$ when $2^n > n > 0$.

Theorem 3.1.5. $\lim_{n\to\infty} \frac{1}{n} = 0$.

Proof Analysis. We will be given $\varepsilon > 0$ and we must find an $N \in \mathbb{N}$ such that if $n > N$ then $\left|\frac{1}{n} - 0\right| < \varepsilon$. Since $\left|\frac{1}{n} - 0\right| = \frac{1}{n}$, we need to find an $N \in \mathbb{N}$ such that if $n > N$ then $\frac{1}{n} < \varepsilon$. Solving the inequality $\frac{1}{n} < \varepsilon$ for n, we see that we must have that $n > \frac{1}{\varepsilon}$. So if we take $N > \frac{1}{\varepsilon}$, then we will be able to prove the desired result. We can now compose a logically correct proof using proof strategy 3.1.2 as a guide.

Proof. Let $\varepsilon > 0$. Let $N > \frac{1}{\varepsilon}$ be a natural number. Let $n > N$. We prove $\left|\frac{1}{n} - 0\right| < \varepsilon$ as follows:

$$\begin{aligned}
\left|\frac{1}{n} - 0\right| &= \left|\frac{1}{n}\right| && \text{by algebra} \\
&= \frac{1}{n} && \text{because } \frac{1}{n} > 0 \\
&< \frac{1}{N} && \text{because } n > N \\
&< \frac{1}{\frac{1}{\varepsilon}} && \text{because } N > \frac{1}{\varepsilon} \\
&= \varepsilon && \text{by algebra.}
\end{aligned}$$

Therefore, $\left|\frac{1}{n} - 0\right| < \varepsilon$. □

Theorem 3.1.6. $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Proof Analysis. We will be given $\varepsilon > 0$ and we must find an $N \in \mathbb{N}$ such that if $n > N$ then $\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon$. Since $\left|\frac{1}{\sqrt{n}} - 0\right| = \frac{1}{\sqrt{n}}$, we need to find an $N \in \mathbb{N}$ such that if $n > N$ then $\frac{1}{\sqrt{n}} < \varepsilon$. Solving the inequality $\frac{1}{\sqrt{n}} < \varepsilon$ for n, we see that we must have that $n > \frac{1}{\varepsilon^2}$. So if we take $N > \frac{1}{\varepsilon^2}$, then we will be able to prove the desired result. We now compose a proof using proof strategy 3.1.2 as a guide.

Proof. Let $\varepsilon > 0$. Let $N > \frac{1}{\varepsilon^2}$ be a natural number. Let $n > N$. Thus, $\frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}}$. We prove that $\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon$ as follows:

$$\begin{aligned}
\left|\frac{1}{\sqrt{n}} - 0\right| &= \left|\frac{1}{\sqrt{n}}\right| && \text{by algebra} \\
&= \frac{1}{\sqrt{n}} && \text{because } \frac{1}{\sqrt{n}} > 0 \\
&< \frac{1}{\sqrt{N}} && \text{because } n > N \\
&< \frac{1}{\sqrt{\frac{1}{\varepsilon^2}}} && \text{because } N > \frac{1}{\varepsilon^2} \\
&= \varepsilon && \text{by algebra.}
\end{aligned}$$

Therefore, $\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon$. □

Theorem. $\lim\limits_{n\to\infty} 1 + \frac{1}{2^n} = 1$.

Proof Analysis. We will be given $\varepsilon > 0$ and we must find an $N \in \mathbb{N}$ such that if $n > N$ then $\left|1 + \frac{1}{2^n} - 1\right| < \varepsilon$. Since $\left|1 + \frac{1}{2^n} - 1\right| = \frac{1}{2^n}$, we need to find an $N \in \mathbb{N}$ such that if $n > N$ then $\frac{1}{2^n} < \varepsilon$. Solving the inequality $\frac{1}{2^n} < \varepsilon$ for n is difficult and so, we take a different approach. It is easy to show by induction (on n) that $n < 2^n$ and thus, $\frac{1}{2^n} < \frac{1}{n}$ when $n \geq 1$. Thus, we will solve the inequality $\frac{1}{n} < \varepsilon$ for n we obtain $n > \frac{1}{\varepsilon}$. Therefore, if we take $N > \frac{1}{\varepsilon}$, then we will be able to prove the desired result. We now compose a proof using proof strategy 3.1.2 as a guide.

Proof. Let $\varepsilon > 0$. Let $N > \frac{1}{\varepsilon}$. Let $n > N$ be a natural number. We prove that $\left|(1 + \frac{1}{2^n}) - 1\right| < \varepsilon$

as follows:

$$\begin{aligned}
\left|\left(1+\frac{1}{2^n}\right)-1\right| &= \left|\frac{1}{2^n}\right| && \text{by algebra} \\
&= \frac{1}{2^n} && \text{because } \frac{1}{2^n} > 0 \\
&< \frac{1}{n} && \text{because } n < 2^n \\
&< \frac{1}{N} && \text{because } n > N \\
&< \frac{1}{\frac{1}{\varepsilon}} && \text{because } N > \frac{1}{\varepsilon} \\
&= \varepsilon && \text{by algebra.}
\end{aligned}$$

Therefore, $\left|1+\frac{1}{2^n}-1\right| < \varepsilon$. □

In the above theorems, we were able to find the required N without much difficulty. Specific sequences with more complicated definitions may require more work to find N. This work can be reduced by correctly applying the Quotient Principles of Inequality 3.1.3.

Theorem. $\lim\limits_{n\to\infty} \frac{2n+3}{3n+5} = \frac{2}{3}$.

Proof Analysis. We will be given $\varepsilon > 0$ and we must find an $N \in \mathbb{N}$ such that if $n > N$ then $\left|\frac{2n+3}{3n+5} - \frac{2}{3}\right| < \varepsilon$. We see that $\left|\frac{2n+3}{3n+5} - \frac{2}{3}\right| = \left|\frac{-1}{9n+15}\right| = \frac{1}{9n+15}$, because $9n+15 > 0$ for $n \in \mathbb{N}$. So, we need to find a natural number N such that if $n > N$ then $\frac{1}{9n+15} < \varepsilon$. One could now solve the inequality $\frac{1}{9n+15} < \varepsilon$ for n, but we take an easier approach. Since $\frac{1}{9n+15} < \frac{1}{9n}$, we find a natural number N such that if $n > N$ then $\frac{1}{9n} < \varepsilon$. Now, solving the inequality $\frac{1}{9n} < \varepsilon$ for n we obtain $n > \frac{1}{9\varepsilon}$. So, if we take $N > \frac{1}{9\varepsilon}$, then we will be able to prove the desired result. We now compose a proof using proof strategy 3.1.2 as a guide.

Proof. Let $\varepsilon > 0$. Let $N > \frac{1}{9\varepsilon}$ be a natural number. Let $n > N$. Thus,

$$\begin{aligned}
\left|\frac{2n+3}{3n+5} - \frac{2}{3}\right| &= \left|\frac{-1}{9n+15}\right| && \text{by algebra.} \\
&= \frac{1}{9n+15} && \text{because } 9n+15 > 0. \\
&< \frac{1}{9n} && \text{because } 9n < 9n+15. \\
&< \frac{1}{9\frac{1}{9\varepsilon}} && \text{because } n > N > \frac{1}{9\varepsilon} \\
&= \varepsilon && \text{by algebra.}
\end{aligned}$$

Therefore, $\left|\frac{2n+3}{3n+5} - \frac{2}{3}\right| < \varepsilon$ and this completes the proof. □

Remark 3.1.7. Suppose that you have a polynomial in n, say $cn^k + 6n^3 - 10$, with highest power n^k with positive coefficient $c > 0$ and some negative coefficients in the lower powers. To find an $s > 0$ such that $sn^k \leq cn^k + 6n^3 - 10$ for all large values of n, you can use any s such that $0 < s < c$.

Example 3.1.8. Given $\varepsilon > 0$, find N such that for all $n > N$, we have that $\left|\frac{n^2+2n-3}{n^2-n-5} - 1\right| < \varepsilon$.

Solution. Let $\varepsilon > 0$. We must find an N such that if $n > N$ then $\left|\frac{n^2+2n-3}{n^2-n-5} - 1\right| < \varepsilon$. By algebra, we obtain

$$\left|\frac{n^2+2n-3}{n^2-n-5} - 1\right| = \left|\frac{3n+2}{n^2-n+5}\right|$$

We see that $3n+2 > 0$ and $n^3 - n^2 + 5 > 0$ when $n \geq 1$. Thus, when $n \geq 1$, we have

$$\left|\frac{3n+2}{n^2-n+5}\right| = \frac{3n+2}{n^2-n+5}.$$

We now need to find an $N \geq 1$ such that if $n > N$ then $\frac{3n+2}{n^2-n+5} < \varepsilon$. Solving the inequality $\frac{3n+2}{n^2-n+5} < \varepsilon$ for n is difficult. So we shall take a different approach by taking advantage of the Quotient Principles of Inequality 3.1.3. First, we will get a real number $b > 0$ such that

$$3n + 2 \leq bn \text{ for all "large" } n \in \mathbb{N}. \tag{3.2}$$

Then we will get a real number $s > 0$ such that

$$sn^2 \leq n^2 - n + 5 \text{ for all "large" } n \in \mathbb{N}. \tag{3.3}$$

To find the b in (3.2), notice that $3n + 2 \leq 3n + 2n = 5n$ for all $n \in \mathbb{N}$. So, we shall let $b = 5$. To find s such that $sn^2 \leq n^2 - n + 5$ for large values of n we must have $0 < s < 1$ (see Remark 3.1.7). Let us try $s = \frac{1}{2}$. We must find m so that $\frac{1}{2}n^2 \leq n^2 - n + 5$ for all $n \geq m$. First, observe that

$$n^2 - n \leq n^2 - n + 5.$$

So we just need that $\frac{1}{2}n^2 \leq n^2 - n$, which is equivalent to $0 \leq n^2 - 2n = n(n-2)$. The latter inequality holds for all $n \geq 2$. Therefore, for all $n \geq 2$ we have that

$$\left|\frac{3n+2}{n^2-n+5}\right| = \frac{3n+2}{n^2-n+5} \leq \frac{5n}{\frac{1}{2}n^2} = \frac{10}{n}.$$

Solving the inequality $\frac{10}{n} < \varepsilon$ for n, we see that we must have that $N > \max\{\frac{10}{\varepsilon}, 2\}$.

Theorem. $\lim\limits_{n\to\infty} \frac{n^2+2n-3}{n^2-n-5} = 1.$

Proof. Let $\varepsilon > 0$ be arbitrary. Let $N > \max\{\frac{6}{\varepsilon}, 3\}$. Let $n > N$. Since $n > 3$, we have that

$$\left|\frac{n^2+2n-3}{n^2-n-5} - 1\right| = \frac{3n+2}{n^2-n+5},$$

because $3n+2 > 0$ and $n^3 - n^2 + 5 > 0$. In addition, as $n > 2$, we have that that $3n + 2 \leq 5n$ and $\frac{1}{2}n^2 \leq n^2 - n + 5$. Therefore, because $n > N > \max\{\frac{10}{\varepsilon}, 2\}$, we have that

$$\left|\frac{n^2+2n-3}{n^2-n-5} - 1\right| = \frac{3n+2}{n^2-n+5} \leq \frac{5n}{\frac{1}{2}n^2} = \frac{10}{n} < \frac{10}{N} < \frac{10}{\frac{10}{\varepsilon}} = \varepsilon.$$

This completes the proof. □

Example 3.1.9. Given $\varepsilon > 0$, find N such that for all $n > N$, we have that $\left|\frac{4n^2+3}{5n^2-2n} - \frac{4}{5}\right| < \varepsilon$.

Solution. Let $\varepsilon > 0$. We must find an N such that if $n > N$ then $\left|\frac{n^2+2n-3}{n^2-n-5} - 1\right| < \varepsilon$. By algebra, we obtain

$$\left|\frac{4n^2+3}{5n^2-2n} - \frac{4}{5}\right| = \left|\frac{8n+15}{25n^2-10n}\right|$$

We see that $8n+15 > 0$ when $n \geq 1$, and $25n^2 - 10n > 0$ when $n \geq 1$. Thus, when $n \geq 1$, we have

$$\left|\frac{4n^2+3}{5n^2-2n} - \frac{4}{5}\right| = \frac{8n+15}{25n^2-10n}.$$

We now need to find an $N \geq 1$ such that if $n > N$ then $\frac{8n+15}{25n^2-10n} < \varepsilon$. Solving the inequality $\frac{8n+15}{25n^2-10n} < \varepsilon$ for n is difficult. So we shall take a different approach by taking advantage of the Quotient Principles of Inequality 3.1.3. First, we will get a real number $b > 0$ such that

$$8n + 15 \leq bn \text{ for all “large” } n \in \mathbb{N}. \tag{3.4}$$

Then we will get a real number $s > 0$ such that

$$sn^2 \leq 25n^2 - 10n \text{ for all “large” } n \in \mathbb{N}. \tag{3.5}$$

To find the b in (3.4), notice that $8n + 15 \leq 8n + 15n = 23n$ for all $n \in \mathbb{N}$. So, we shall let $b = 23$. To find s such that $sn^2 \leq 25n^2 - 10n$ for large values of n we must have $0 < s < 25$ (see Remark 3.1.7). Let us try $s = 20$. We must find m so that $20n^2 \leq 25n^2 - 10n$ for all $n \geq m$. Observe that $20n^2 \leq 25n^2 - 10n$ is equivalent to $0 \leq 5n^2 - 10n = 5n(n-2)$. The latter inequality holds for all $n \geq 2$. Therefore, for all $n \geq 2$ we have that

$$\left|\frac{4n^2+3}{5n^2-2n} - \frac{4}{5}\right| = \frac{8n+15}{25n^2-10n} \leq \frac{23n}{20n^2} = \frac{23}{20n}.$$

Solving the inequality $\frac{23}{20n} < \varepsilon$ for n, we see that we must have that $N > \max\{\frac{23}{20\varepsilon}, 2\}$.

Theorem. $\lim\limits_{n\to\infty} \frac{n^2+2n-3}{n^2-n-5} = \frac{4}{5}$.

Proof. Let $\varepsilon > 0$ be arbitrary. Let $N > \max\{\frac{23}{20\varepsilon}, 2\}$. Let $n > N$. Since $n > 2$, we have that

$$\left|\frac{4n^2+3}{5n^2-2n} - \frac{4}{5}\right| = \frac{8n+15}{25n^2-10n}.$$

because $8n + 15 > 0$ and $25n^2 - 10n > 0$. In addition, as $n > 2$, we have that that $8n + 15 \leq 23n$ and $20n^2 \leq 25n^2 - 10n$. Therefore, because $n > N > \max\{\frac{23}{20\varepsilon}, 2\}$, we have that

$$\left|\frac{4n^2+3}{5n^2-2n} - \frac{4}{5}\right| = \frac{8n+15}{25n^2-10n} \leq \frac{23n}{20n^2} = \frac{23}{20n} < \frac{23}{20N} < \frac{23}{20\frac{23}{20\varepsilon}} = \varepsilon.$$

This completes the proof. □

Suppose in a proof that you are assuming that a given sequence converges. Our next strategy will be useful when dealing with such an assumption.

Assumption Strategy 3.1.10. Suppose you are assuming that $\lim\limits_{n\to\infty} s_n = \ell$. Then for any $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $|s_n - \ell| < \varepsilon$ for all $n > N$.

Thus, in a proof, suppose that you are assuming that $\lim\limits_{n\to\infty} s_n = \ell$. Using assumption strategy 3.1.10 you can conclude that for any positive value $v > 0$ there is an N such that for all $n > N$ we have that $|s_n - \ell| < v$. We shall express this observation as “we can make $|s_n - \ell|$ as small as we want.” We shall now apply this idea to prove the following theorem.

Theorem 3.1.11. Suppose $\lim_{n\to\infty} s_n = \ell$ where $\ell > 0$ and $s_n > 0$ for all $n \geq 1$. Then $\lim_{n\to\infty} \sqrt{s_n} = \sqrt{\ell}$.

Proof Analysis. In a proof of the above theorem, we shall be assuming that $\lim_{n\to\infty} s_n = \ell$ and we must prove that $\lim_{n\to\infty} \sqrt{s_n} = \sqrt{\ell}$. How can one apply proof strategy 3.1.2 and assumption strategy 3.1.10 to find such a proof? First of all, our proof will need to have the following logical structure:

Assume $\lim_{n\to\infty} s_n = \ell$.
Let $\varepsilon > 0$ be an arbitrary real number.
  Let $N =$ (the natural number you found).
    Let $n > N$ be an arbitrary natural number.
      Prove $\left|\sqrt{s_n} - \sqrt{\ell}\right| < \varepsilon$.

We must find a natural number N such that if $n > N$, then $\left|\sqrt{s_n} - \sqrt{\ell}\right| < \varepsilon$. We shall use the assumption $\lim_{n\to\infty} s_n = s$ to find the desired N. Here is the basic plan that we will apply to get N.

> *Using algebra and properties of inequality on the expression* $\left|\sqrt{s_n} - \sqrt{\ell}\right|$*, we "extract out" a larger value that contains* $|s_n - \ell|$ *and no other occurrences of* s_n*.*

Since $\lim_{n\to\infty} s_n = \ell$, we can make $|s_n - \ell|$ "as small as we want." We should then be able to make $\left|\sqrt{s_n} - \sqrt{\ell}\right| < \varepsilon$ and find the N that we need. Let us now execute this plan! First we start with $\left|\sqrt{s_n} - \sqrt{\ell}\right|$ and extract out $|s_n - \ell|$ as follows:

$$\begin{aligned}
\left|\sqrt{s_n} - \sqrt{\ell}\right| &= \left|\frac{(\sqrt{s_n} - \sqrt{\ell})}{1}\frac{(\sqrt{s_n} + \sqrt{\ell})}{(\sqrt{s_n} + \sqrt{\ell})}\right| && \text{rationalizing the numerator.}\\
&= \left|\frac{s_n - \ell}{\sqrt{s_n} + \sqrt{\ell}}\right| && \text{by algebra.}\\
&= \frac{|s_n - \ell|}{\sqrt{s_n} + \sqrt{\ell}} && \text{because } \sqrt{s_n} + \sqrt{\ell} > 0.\\
&< \frac{|s_n - \ell|}{\sqrt{\ell}} && \text{because } \sqrt{\ell} < \sqrt{s_n} + \sqrt{\ell}.
\end{aligned}$$

Thus, we started with $\left|\sqrt{s_n} - \sqrt{\ell}\right|$ and we were able to extract out the larger value $\frac{|s_n-\ell|}{\sqrt{\ell}}$ that contains $|s_n - \ell|$ and no other occurrences of s_n. Hence, we have that $\left|\sqrt{s_n} - \sqrt{\ell}\right| < \frac{|s_n-\ell|}{\sqrt{\ell}}$. Consequently, if $\frac{|s_n-\ell|}{\sqrt{\ell}} < \varepsilon$, then we will have that $\left|\sqrt{s_n} - \sqrt{\ell}\right| < \varepsilon$. How small must $|s_n - \ell|$ be in order to ensure that $\frac{|s_n-\ell|}{\sqrt{\ell}} < \varepsilon$? To answer this question, we just solve this latter inequality for $|s_n - \ell|$ to obtain $|s_n - \ell| < \varepsilon\sqrt{\ell}$. Hence, we need an N so that $|s_n - \ell| < \varepsilon\sqrt{\ell}$ when $n > N$. Since $\lim_{n\to\infty} s_n = \ell$, there is such an N. This is the value for N that we will use in our proof.

Proof of Theorem 3.1.11. Assume that $\lim_{n\to\infty} s_n = \ell$ where $\ell > 0$ and $s_n > 0$ for all $n \geq 1$. We shall prove that $\lim_{n\to\infty} \sqrt{s_n} = \sqrt{\ell}$. To do this, let $\varepsilon > 0$. Since $\lim_{n\to\infty} s_n = \ell$, there is such an N such that

($\star$) $|s_n - \ell| < \varepsilon\sqrt{\ell}$ for all $n > N$. Let $n > N$. We will now prove that $\left|\sqrt{s_n} - \sqrt{\ell}\right| < \varepsilon$ as follows:

$$\begin{aligned}
\left|\sqrt{s_n} - \sqrt{\ell}\right| &= \left|\frac{(\sqrt{s_n} - \sqrt{\ell})}{1}\frac{(\sqrt{s_n} + \sqrt{\ell})}{(\sqrt{s_n} + \sqrt{\ell})}\right| && \text{rationalizing the numerator.}\\
&= \left|\frac{s_n - \ell}{\sqrt{s_n} + \sqrt{\ell}}\right| && \text{by algebra.}\\
&= \frac{|s_n - \ell|}{\sqrt{s_n} + \sqrt{\ell}} && \text{because } \sqrt{s_n} + \sqrt{\ell} > 0.\\
&< \frac{|s_n - \ell|}{\sqrt{\ell}} && \text{because } \sqrt{\ell} < \sqrt{s_n} + \sqrt{\ell}.\\
&< \frac{\varepsilon\sqrt{\ell}}{\sqrt{\ell}} && \text{by } (\star).\\
&= \varepsilon && \text{by algebra.}
\end{aligned}$$

Therefore, $\left|\sqrt{s_n} - \sqrt{\ell}\right| < \varepsilon$ and this completes the proof. □

Theorem 3.1.12 (Uniqueness of the Limit)**.** If the sequence $\langle s_n\rangle$ converges, then there is only one limit of the sequence.

Proof. Suppose the sequence $\langle s_n\rangle$ converges. To prove that there is only one limit of this sequence, suppose that ℓ and ℓ' are both limits of the sequence $\langle s_n\rangle$. We shall prove that $\ell = \ell'$. For a contradiction, assume that $\ell \neq \ell'$. Thus, we conclude that $\varepsilon = |\ell - \ell'| > 0$. Since $\langle s_n\rangle$ converges to ℓ there is an $N \in \mathbb{N}$ such that for all $n > N$, $|s_n - \ell| < \frac{\varepsilon}{2}$. Also, since $\langle s_n\rangle$ converges to ℓ' there is an $N' \in \mathbb{N}$ such that for all $n > N'$, $|s_n - \ell'| < \frac{\varepsilon}{2}$. Therefore, for all $n > \max\{N, N'\}$ we have that

$$\left|\ell - \ell'\right| = \left|(\ell - s_n) + (s_n - \ell')\right| \leq |\ell - s_n| + \left|s_n - \ell'\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $|\ell - \ell'| < \varepsilon$. But this contradicts our conclusion that $\varepsilon = |\ell - \ell'|$. This contradiction completes the proof of the theorem. □

Theorem 3.1.13. If $\langle s_n\rangle$ is a sequence and $\ell \in \mathbb{R}$, then $\lim\limits_{n\to\infty} s_n = \ell$ if and only if $\lim\limits_{n\to\infty}(s_n - \ell) = 0$.

Proof. Assume that $\lim\limits_{n\to\infty} s_n = \ell$. We shall prove that $\lim\limits_{n\to\infty}(s_n - \ell) = 0$. To do this, let $\varepsilon > 0$. Since $\lim\limits_{n\to\infty} s_n = \ell$, there is an $N \in \mathbb{N}$ where $|s_n - \ell| < \varepsilon$ for all $n > N$. Let $n > N$. Thus $|(s_n - \ell) - 0| = |s_n - \ell| < \varepsilon$ and thus $\lim\limits_{n\to\infty}(s_n - \ell) = 0$. The converse follows similarly. □

Suppose that you are assuming that a given sequence converges and you need to prove that another sequence converges. The assumption strategy 3.1.10 will be useful when dealing with such a proof. For example, many times we will be assuming that $\lim\limits_{n\to\infty} s_n = \ell$ and will be working with $\varepsilon > 0$. Using assumption strategy 3.1.10 we can conclude for any positive $\varepsilon' < \varepsilon$ (for example, $\varepsilon' = \frac{\varepsilon}{2}$) that there is an N' such that for all $n > N'$ we we have $|s_n - \ell| < \varepsilon'$.

Theorem 3.1.14. Let $\langle s_n\rangle$, $\langle a_n\rangle$ be sequences and let $\ell \in \mathbb{R}$. If

(1) $|s_n - \ell| \leq k\,|a_n|$ for all $n \geq m$, where $k > 0$ and $m \in \mathbb{N}$,

(2) $\lim\limits_{n\to\infty} a_n = 0$,

then $\lim\limits_{n\to\infty} s_n = \ell$.

Proof Analysis. We will assume that $|s_n - \ell| \le k\,|a_n|$ for all $n \ge m$ where $k > 0$ and $m \in \mathbb{N}$. We also assume that $\lim_{n\to\infty} a_n = 0$ and thus, we can make $|a_n| = |a_n - 0|$ 'as small as we want.' We will be given $\varepsilon > 0$ and we must find an $N \in \mathbb{N}$ such that if $n > N$ then $|s_n - \ell| < \varepsilon$. Since $|s_n - \ell| \le k\,|a_n|$ when $n \ge m$, we need to find an $N \ge m$ such that if $n > N$ then $k\,|a_n| < \varepsilon$. Solving this latter inequality for $|a_n|$, we see that we must have that $|a_n| < \frac{\varepsilon}{k}$. Since $\lim_{n\to\infty} a_n = 0$, there is an N' such that whenever $n > N'$ we have $|a_n - 0| = |a_n| < \frac{\varepsilon}{k}$. Thus, we will use $N = \max\{N', m\}$. This will ensure that when $n > N$ we will have that $n > N'$ and $n > m$.

Proof. Assuming (1) and (2) we shall prove that $\lim_{n\to\infty} s_n = \ell$. To do this, let $\varepsilon > 0$. By (2) we have that $\lim_{n\to\infty} a_n = 0$. Thus, there is an $N' \in \mathbb{N}$ such that $(\star)$ $|a_n - 0| = |a_n| < \frac{\varepsilon}{k}$ for all $n > N'$. By (1), we have that $(\star\star)$ $|s_n - \ell| \le k\,|a_n|$ for all $n \ge m$. Let $N = \max\{m, N'\}$. Let $n > N$. Thus,

$$\begin{aligned} |s_n - \ell| &\le k\,|a_n| && \text{by } (\star\star) \text{ because } n > N \ge m. \\ &< k\left(\frac{\varepsilon}{k}\right) && \text{by } (\star) \text{ because } n > N \ge N'. \\ &= \varepsilon && \text{by algebra.} \end{aligned}$$

Therefore $|s_n - \ell| < \varepsilon$. This completes the proof of the theorem. □

Corollary 3.1.15. Let x be a such that $|x| < 1$. Then $\lim_{n\to\infty} x^n = 0$.

Proof. Let x be such that $|x| < 1$. If $x = 0$, then clearly $\lim_{n\to\infty} x^n = 0$. Assume that $0 < |x| < 1$. Since $\frac{1}{|x|} > 1$, there is a $c > 0$ such that $1 + c = \frac{1}{|x|}$. By Bernoulli's inequality (Exercise 5-page 28), $(1+c)^n \ge 1 + nc$, for every $n \in \mathbb{N}$. Hence, $\frac{1}{|x|^n} = (1+c)^n \ge 1 + nc > nc$, for all $n \in \mathbb{N}$. So

$$|x^n - 0| = |x|^n < \frac{1}{nc},$$

for all $n \ge 1$. As $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\frac{1}{c} > 0$, Theorem 3.1.14 implies that $\lim_{n\to\infty} x^n = 0$. □

The next corollary can be useful for proving that a particular sequence $\langle u_n \rangle$ converges to r.

Corollary 3.1.16. Let $\langle s_n \rangle$ be a sequence such that $\lim_{n\to\infty} s_n = \ell$ for a real number ℓ. Let $\langle u_n \rangle$ be another sequence and a real number r satisfying $|u_n - r| \le k\,|s_n - \ell|$ for all $n \ge m$, for some $k > 0$ and $m \in \mathbb{N}$. Then $\lim_{n\to\infty} u_n = r$.

Proof. By Theorem 3.1.13 we have that $\lim_{n\to\infty}(s_n - \ell) = 0$. We are given that $|u_n - r| \le k\,|s_n - \ell|$ for all $n \ge m$, for some $k > 0$ and $m \in \mathbb{N}$. Theorem 3.1.14 now implies that $\lim_{n\to\infty} u_n = r$. □

Remark 3.1.17. Let $\langle s_n \rangle$ be sequence and let $\ell \in \mathbb{R}$. What does it mean to say that the sequence $\langle s_n \rangle$ does **not** converge to ℓ? Taking the negation of the logical form (3.1), we conclude that the statement "the sequence $\langle s_n \rangle$ does **not** converge to ℓ" means the following:

There is an $\varepsilon > 0$ such that for all $N \in \mathbb{N}$ there is an $n > N$ such that $|s_n - \ell| \ge \varepsilon$.

Neighborhoods

Definition 3.1.18. Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. The open interval $(x - \varepsilon, x + \varepsilon)$, centered at x, is called a **neighborhood** of x.

Let x be a real number and let U_ε be the neighborhood $(x - \varepsilon, x + \varepsilon)$ of x where $\varepsilon > 0$. Then for any real number s we have that $s \in U_\varepsilon$ if and only if $|s - x| < \varepsilon$. We shall write U to denote a neighborhood of x when it is not important to specify ε. The following theorem just states that the notion of convergence can be expressed in terms of neighborhoods (see Figure 3.3).

Theorem 3.1.19. Let $\langle s_n \rangle$ be a sequence and let ℓ be a real number. Then the following are equivalent:

1. The sequence $\langle s_n \rangle$ converges to ℓ.
2. For every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n > N$ then $|s_n - \ell| < \varepsilon$.
3. For every neighborhood U of ℓ there is an $N \in \mathbb{N}$ so that for all $n \in \mathbb{N}$, if $n > N$ then $s_n \in U$.

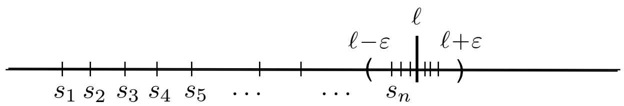

Figure 3.3: For all $n > N$ we have $|s_n - \ell| < \varepsilon$.

Corollary 3.1.20. Let $\langle s_n \rangle$ be a sequence of distinct points and suppose that $\langle s_n \rangle$ converges to ℓ. Then every neighborhood of ℓ contains an infinite number of points from the sequence $\langle s_n \rangle$.

Proof. Let U be any neighborhood of ℓ. Since the sequence $\langle s_n \rangle$ converges to ℓ, Theorem 3.1.19 states that there is an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n > N$ then $s_n \in U$. Therefore, an infinite number of points from the sequence $\langle s_n \rangle$ are in U. $\square$

Remark. Given a sequence $\langle s_n \rangle$ that converges to ℓ. Theorem 3.1.19 implies that for every neighborhood U of ℓ there exists an $N \in \mathbb{N}$ such that $s_n \in U$ for all $n > N$. We can say in this case that the sequence $\langle s_n \rangle$ is *eventually* in U.

Lemma 3.1.21. Assume that $D \subseteq \mathbb{R}$ is dense in $\mathbb{R}$. Let x be any real number. Then there is a sequence $\langle d_n \rangle$ that converges to x where $d_n \in D$ for all $n \geq 1$.

Proof. Since $D \subseteq \mathbb{R}$ is dense in $\mathbb{R}$, for each $n \in \mathbb{N}$ there is a $d_n \in D$ such that d_n is in the interval $(x - \frac{1}{n}, x + \frac{1}{n})$ and so, $|x - d_n| < \frac{1}{n}$. Since $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, Theorem 3.1.14 implies that $\lim\limits_{n\to\infty} d_n = x$. $\square$

Bounded Sequences

Definition 3.1.22. A sequence $\langle s_n \rangle$ is **bounded** if there is are real numbers a and b such that $a \leq s_n \leq b$ for all $n \in \mathbb{N}$.

Remark 3.1.23. A sequence $\langle s_n \rangle$ is bounded if and only if there is an $M > 0$ so that $|s_n| \leq M$ for all $n \in \mathbb{N}$ (see Theorem 2.2.2).

Let $\langle s_n \rangle$ be a sequence. Suppose there is an $B > 0$ and an N such that $|s_n| \leq B$ for all $n > N$. Then one can conclude that the entire sequence is bounded; that is, that there is an $M > 0$ such that $|s_n| \leq M$ for all $n \geq 1$. The following example illustrates this fact.

Example 3.1.24. Let $\langle s_n \rangle$ be the sequence where $s_n = 2 + \frac{1}{n}$ where $n \geq 1$. Let $B = s_{10} = 2 + \frac{1}{10}$. Notice that for all $n > 10$ we have $s_n \leq B$. But all of the values $s_1, \ldots, s_{10}$ are greater than B. Let $M = \max\{s_1, \ldots, s_{10}, B\} = 3$. We now have that $s_n \leq 3$ for all $n \geq 1$. Thus, the sequence $\langle s_n \rangle$ is bounded.

Theorem 3.1.25. Let $\langle s_n \rangle$ be a convergent sequence. Then the sequence $\langle s_n \rangle$ is bounded.

Proof. Suppose that $\lim\limits_{n\to\infty} s_n = \ell$. Thus, for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $|s_n - \ell| < \varepsilon$ for all $n > N$. So, lets take $\varepsilon = 1$ and let $N \in \mathbb{N}$ be so that $|s_n - \ell| < 1$ for all $n > N$. Thus,

$$|s_n| - |\ell| \leq |s_n - \ell| < 1$$

for all $n > N$. Hence, $|s_n| < |\ell| + 1$ for all $n > N$. Let

$$M = \max\{|s_1|, \ldots, |s_N|, |\ell| + 1\}.$$

Therefore, $\langle s_n \rangle$ is a bounded sequence. $\square$

Remark 3.1.26. If a sequence $\langle s_n \rangle$ is unbounded, then the sequence $\langle s_n \rangle$ diverges.

Example 3.1.27. Consider the sequence $\langle s_n \rangle$ where $s_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Now imagine n to be very large and write

$$\begin{aligned} s_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} \\ &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots + \frac{1}{n} \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right) + \cdots + \frac{1}{n} \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{n}. \end{aligned}$$

It is clear that by taking n sufficiently large we can introduce as many $\frac{1}{2}$'s, in the this sum, as we wish. Therefore, the sequence $\langle s_n \rangle$ is unbounded and so, it diverges.

Remark 3.1.28. Suppose that the sequence $\langle s_n \rangle$ converges to ℓ. Then for $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$|s_n - \ell| < \varepsilon \text{ for all } n > N.$$

So, for any $K > N$ we can also conclude that

$$|s_n - \ell| < \varepsilon \text{ for all } n > K.$$

Exercises 3.1

1. Using Axiom O4 of the ordered field axioms on page 30 and Theorem 1.1.18 on page 11, prove the three properties of inequality stated in 3.1.3.
2. Let $a \in \mathbb{R}$. Prove that the sequence $\left\langle a + (-1)^n \frac{2n+1}{n} \right\rangle$ is bounded.
3. Let $k > 0$. Use Definition 3.1.1 to prove that $\lim_{n\to\infty} \frac{k}{n} = 0$.
4. Use Definition 3.1.1 to prove that $\lim_{n\to\infty} \frac{n+1}{n+2} = 1$.
5. Use Definition 3.1.1 to prove that $\lim_{n\to\infty} \frac{3n}{n+2} = 3$.
6. Use Definition 3.1.1 to prove that $\lim_{n\to\infty} \frac{6n-7}{3n-2} = 2$.
7. Use Definition 3.1.1 to prove that $\lim_{n\to\infty} \frac{6n-7}{2n-7} = 3$.
8. Prove that the limits given in Exercises 4–7 hold by applying Theorems 3.1.14 and 3.1.5.
9. Prove Theorem 3.1.11 using Corollary 3.1.16.
10. Let $\langle s_n \rangle$ be a convergent sequence. Suppose $\lim_{n\to\infty} s_n = \ell$ and let $c \in \mathbb{R}$ be a constant. Prove that $\lim_{n\to\infty} c + s_n = c + \ell$.
11. Let $\langle s_n \rangle$ be a convergent sequence. Suppose $\lim_{n\to\infty} s_n = \ell$ and let $c \in \mathbb{R}$ be a nonzero constant. Prove that $\lim_{n\to\infty} (cs_n) = c\ell$.
12. Suppose that $\lim_{n\to\infty} s_n = \ell$. Prove that $\lim_{n\to\infty} |s_n| = |\ell|$.
13. Suppose that $\lim_{n\to\infty} |s_n| = 0$. Prove that $\lim_{n\to\infty} s_n = 0$.

14. Suppose that $\lim_{n\to\infty} s_n = \ell$ and $|s_n| \le M$ for all $n \ge 1$, where $M > 0$. Prove that $\lim_{n\to\infty} {s_n}^2 = \ell^2$.

15. Let $c \in \mathbb{R}$ be constant. Prove that $\lim_{n\to\infty} c = c$.

16. Let $\langle x_n\rangle$ and $\langle y_n\rangle$ be two convergent sequences. Prove that there exists an $M > 0$ such that $|x_n| \le M$ and $|y_n| \le M$ for all $n \ge 1$.

17. Let $\langle s_n\rangle$ a convergent sequence. Suppose that $\lim_{n\to\infty} s_n = \ell$. Prove that there exists an $M > 0$ such that $|s_n + \ell| \le M$ for all $n \ge 1$.

18. Use Theorems 3.1.14 and 3.1.5 to prove that $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$.

19. Use Theorem 3.1.14 and Theorem 3.1.6 to prove that $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0$.

20. Let $\langle s_n\rangle$ a convergent sequence. Suppose that $\lim_{n\to\infty} s_n = \ell$. Prove that $\lim_{n\to\infty} s_n^2 = \ell^2$.

21. Let x be a such that $|x| > 1$. Prove that the sequence $\langle x^n\rangle$ is not bounded.

Exercise Notes: For Exercise 6, observe that $n \le 3n - 2$ when $n \ge 1$. For Exercise 7, observe that $n < |2n - 7|$ when $n > 7$. In this case, we would need N to be at least 7 and so, $N > \max\{7, \frac{14}{\varepsilon}\}$ could be used in the proof. For Exercise 19, show that $\left|\sqrt{n+1} - \sqrt{n}\right| = \frac{1}{\sqrt{n+1}+\sqrt{n}} \le \frac{1}{2\sqrt{n}}$. For Exercise 21, see the proof of Corollary 3.1.15.

3.2 Limit Theorems for Sequences

What are Limit Theorems? A Limit Theorem states that if you know the limits of some given sequences, then you can determine the limit of a new sequence that is related to the given sequences. Limit theorems have the form:

Theorem. Suppose we are given that $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$. Then one can evaluate the limit of a new sequence, say $\lim_{n\to\infty} u_n = u$, which is constructed from the given sequences.

How does one prove theorems that have this form? We are assuming that $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$, and we must prove that $\lim_{n\to\infty} u_n = u$.

Proof Diagram 3.2.1. To prove that $\lim_{n\to\infty} u_n = u$, our proof must contain the structure

Assume $\lim_{n\to\infty} s_n = s$.
Assume $\lim_{n\to\infty} t_n = t$.
Let $\varepsilon > 0$ be an arbitrary real number.
    Let $N =$ (the natural number you found).
        Let $n > N$ be an arbitrary natural number.
            Prove $|u_n - u| < \varepsilon$.

Let $\varepsilon > 0$ be given. We must find a natural number N such that if $n > N$ then $|u_n - u| < \varepsilon$. We can use the assumptions $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$, to find the desired N. Here is the basic idea that we will apply to get N.

Using algebra and properties of inequality on the expression $|u_n - u|$, we "extract out" a larger value containing $|s_n - s|$ and $|t_n - t|$, and no other occurrences of s_n or t_n.