Chapter 2

Axioms for the Real Numbers

2.1 $\mathbb{R}$ is an Ordered Field

Real analysis is an branch of mathematics that studies the set $\mathbb{R}$ of real numbers and provides a theoretical foundation for the fundamental principles of the calculus. The main concepts studied are sets of real numbers, functions, limits, sequences, continuity, differentiation, integration and sequences of functions. Among the first topics covered in a typical real analysis course are the ordered field axioms. These axioms form a basis for the algebraic operations and the order properties upon which the calculus is based.

The set of real numbers $\mathbb{R}$ has two binary operations $+$ and $\cdot$ (called addition and multiplication) and a relation $<$ (called 'less than') satisfying the **ordered field axioms**:

- A1. For all $x, y \in \mathbb{R}$, $x + y = y + x$.
- A2. For all $x, y, z \in \mathbb{R}$, $x + (y + z) = (x + y) + z$.
- A3. There is a unique number 0 such that for all $x \in \mathbb{R}$, $x + 0 = x$.
- A4. For all $x \in \mathbb{R}$ there exists a unique $y \in \mathbb{R}$ such that $x + y = 0$. (Recall $y = -x$.)
- M1. For all $x, y \in \mathbb{R}$, $x \cdot y = y \cdot x$.
- M2. For all $x, y, z \in \mathbb{R}$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
- M3. There is a unique number 1 such that $1 \neq 0$ and for all $x \in \mathbb{R}$, $x \cdot 1 = x$.
- M4. For all nonzero $x \in \mathbb{R}$ there exists a unique $y \in \mathbb{R}$ such that $x \cdot y = 1$. (Recall $y = x^{-1}$.)
- D1. For all $x, y, z \in \mathbb{R}$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.
- O1. For all $x, y \in \mathbb{R}$, exactly one of the following relations holds: $x < y$, $y < x$, or $x = y$.
- O2. For all $x, y, z \in \mathbb{R}$, if $x < y$, and $y < z$, then $x < z$.
- O3. For all $x, y, z \in \mathbb{R}$, if $x < y$, then $x + z < y + z$.
- O4. For all $x, y, z \in \mathbb{R}$, if $x < y$ and $z > 0$, then $x \cdot z < y \cdot z$.

Note that A3 implies that $0 + 0 = 0$; and M3 implies that $1 \cdot 1 = 1$. Using the ordered field axioms one can prove our next theorem.

Theorem 2.1.1. For all $x, y, z \in \mathbb{R}$ the following six items hold:

(a) If $x + z = y + z$, then $x = y$.

(b) $x \cdot 0 = 0$.

(c) $(-1) \cdot x = -x$.

(d) $x \cdot y = 0$ if and only if $x = 0$ or $y = 0$.

(e) $x < y$ if and only if $-y < -x$.

(f) If $x < y$ and $z < 0$, then $xz > yz$.

Theorem 2.1.2. Let $x, y \in \mathbb{R}$. Suppose $x \leq y + \varepsilon$ for all $\varepsilon > 0$. Then $x \leq y$.

Proof. Let $x, y \in \mathbb{R}$ and assume that $x \leq y + \varepsilon$ for all $\varepsilon > 0$. We will prove that $x \leq y$. Suppose, for a contradiction, that $x > y$. Thus, $x - y > 0$ and $\varepsilon = \frac{x-y}{2} > 0$. So, $x \leq y + \varepsilon$ by our assumption. Since $\varepsilon = \frac{x-y}{2} < x - y$, we obtain

$$x \leq y + \varepsilon = y + \left(\frac{x-y}{2}\right) < y + (x - y) = x. \tag{2.1}$$

Therefore, from (2.1) we conclude that $x < x$, a contradiction. □

Definition 2.1.3 (Absolute Value)**.** Given a real number x, the **absolute value** of x, denoted by $|x|$, is defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

Theorem 2.1.4 (Basic Properties of Absolute Value)**.** For all $a, x \in \mathbb{R}$, where $a \geq 0$, the following hold:

(a) $0 \leq |x|$, $x \leq |x|$, $-x \leq |x|$, $|-x| = |x|$.

(b) if $|x| = 0$, then $x = 0$.

(c) $|x| \leq a$ if and only if $-a \leq x \leq a$.

(d) $|xy| = |x||y|$.

(e) $|x|^2 = x^2$.

(f) $|x + y| \leq |x| + |y|$ (triangle inequality).

Proof. Items (a)-(e) follow directly from Definition 2.1.3. We shall now prove (f) as follows: Since $|x + y|^2 = (x + y)^2$, we conclude that

$$|x + y|^2 = x^2 + 2xy + y^2 \leq x^2 + |2xy| + y^2 = |x|^2 + 2\,|x|\,|y| + |y|^2 = (|x| + |y|)^2.$$

Thus, $|x + y|^2 \leq (|x| + |y|)^2$. Theorem 1.1.16 implies that $|x + y| \leq |x| + |y|$. □

Theorem 2.1.5 (More Properties of Absolute Value)**.** For all $x, y, k \in \mathbb{R}$, where $k > 0$, we have

(i) $|x| < k$ if and only if $-k < x < k$.

(ii) $|x| > k$ if and only if $x < -k$ or $x > k$.

(iii) $|x| - |y| \leq |x - y|$.

(iv) $|y| - |x| \leq |x - y|$.

(v) $\big|\,|x| - |y|\,\big| \leq |x - y|$ (backward triangle inequality).

Proof. Items (i)-(ii) follow directly from Definition 2.1.3. Note that (iii) and (iv) imply (v), using Definition 2.1.3. We shall prove (iii) and (iv). First we prove (iii). Observe that $|x| = |x - y + y|$. Hence, the triangle inequality implies that

$$|x| = |x - y + y| \leq |x - y| + |y|\,.$$

So, $|x| \le |x-y|+|y|$ and thus, $|x|-|y| \le |x-y|$. Now we prove (iv). Observe that $|y| = |y-x+x|$. Hence, the triangle inequality implies that

$$|y| = |y-x+x| \le |y-x| + |x|\,.$$

Since $|y-x| = |x-y|$, we conclude that $|y| \le |x-y| + |x|$ and thus, $|y|-|x| \le |x-y|$. □

Given a finite nonempty set of real numbers A, we let $\max A$, or $\max(A)$, denote the maximum number in A. Similarly, we define $\min A$, or $\min(A)$, to be the minimum number in A. For example, $\max\{-1,2,\pi,3\} = \pi$ and $\min\{-1,2,\pi,3\} = -1$.

Lemma 2.1.6. Let x, a and b be real numbers. If $a \le x \le b$, then $|x| \le \max\{|a|, |b|\}$.

Proof. Assume that $a \le x \le b$. We shall prove that $|x| \le \max\{|a|, |b|\}$. First suppose that $x \ge 0$. So, because $x \le b$, we conclude that $b \ge 0$. So, $|x| = x$ and $|b| = b$. Since $x \le b$, we see that $|x| \le |b|$ and therefore, $|x| \le \max\{|a|, |b|\}$. Now suppose that $x < 0$. Then, because $a \le x$, we see that $a < 0$. So, $|x| = -x$ and $|a| = -a$. Since $a \le x$, we see that $-x \le -a$ and so, $|x| \le |a|$. Therefore, $|x| \le \max\{|a|, |b|\}$. □

Lemma 2.1.7. Let $a \le x \le b$ and $a \le y \le b$. Then $|x-y| \le |b-a|$.

Proof. Assume (1) $a \le x \le b$ and $a \le y \le b$. Thus, (2) $-b \le -y \le -a$. Adding (1) and (2), we obtain $-(b-a) \le x-y \le (b-a)$. Thus, $|x-y| \le |b-a|$ by Theorem 2.1.4(c). □

Exercises 2.1

1. Prove Theorem 2.1.1.
2. Prove (c) of Theorem 2.1.4.
3. Prove (i)-(iii) of Theorem 2.1.5.
4. Justify the equalities and inequalities used in the proof of Theorem 2.1.4(f) on page 30.
5. Let $a > 0$. Prove that if $|x-a| < a$, then $x > 0$.
6. Suppose that $x > 0$ and $y > 0$. Prove that $x^2 + y^2 < (x+y)^2$.
7. Suppose that $x < 0$ and $y < 0$. Prove that $x^2 + y^2 < (x+y)^2$.
8. Suppose that $x > 0$ and $y > 0$. Prove that $\sqrt{x+y} < \sqrt{x} + \sqrt{y}$.
9. Let $a < x < b$ and $a < y < b$. Prove that $|x-y| < |b-a|$.

Exercise Notes: For Exercise 5, use Theorem 2.1.5(i). For Exercise 8 use proof by contradiction and Theorem 1.1.15.

2.2 The Completeness Axiom

In this section we present the completeness axiom, an assertion that implies there are no gaps or holes in the real number line. We shall see that this axiom is an essential property of the real number system that will allows us to conclude many of the fundamental theorems in the calculus; for example, the Cauchy Convergence Criterion, the Intermediate Value Theorem, and the Fundamental Theorem of Calculus.