Chapter 1

Proof, Sets, and Functions

1.1 Proofs

A review of logic and proof[1] is presented in Appendix C on page 176. The most important proof strategies that are applied in this text can be found in the appendix starting on page 179.

Conjecture + Proof = Theorem

A **conjecture** is a statement that seems plausible but whose truth has not been established. In mathematics one never accepts a conjecture as true until a mathematical **proof** of the conjecture has been given. Once a mathematical proof of the conjecture is produced we then call the conjecture a **theorem.** On the other hand, to show that a conjecture is false one must find a particular assignment of values (an example) making the statement of the conjecture false. Such an assignment is called a **counterexample** to the conjecture.

The Proof Is Completed

It is convenient to have a mark which signals the end of a proof. Mathematicians in the past, would end their proofs with letters Q.E.D., an abbreviation for the Latin expression "quod erat demonstrandum." So in English, we interpret Q.E.D. to mean "that which was to be demonstrated." In current times, mathematicians typically use the symbol $\square$ to let the reader know that the proof has been completed. In this text, we shall do the same.

1.1.1 Important Sets in Mathematics

Certain sets are frequently used in mathematics. The most commonly used ones are the sets of whole numbers, natural numbers, integers, rational and real numbers. These sets will be denoted by the following symbols:

1. $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of natural numbers.
2. $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ is the set of integers.
3. $\mathbb{Q}$ is the set of rational numbers; that is, the set of numbers $r = \frac{a}{b}$ for integers a, b where $b \neq 0$. So, $\frac{3}{2} \in \mathbb{Q}$.
4. $\mathbb{R}$ is the set of real numbers and so, $\pi \in \mathbb{R}$.

[1] MAT 300 covers proof and logic.

In this text, we do **not** consider 0 to be a natural number. For each of the sets $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, we may add '+' or '−' as a superscript. The + (or −) superscript indicates that only the positive (or negative) numbers will be allowed. For example,

1. $\mathbb{Q}^+ = \{x : x \text{ is a positive rational number}\}$.
2. $\mathbb{Z}^- = \{x : x \text{ is a negative integer}\}$.
3. $\mathbb{R}^+ = \{x : x \text{ is a positive real number}\}$.

For sets A and B we write $A \subseteq B$ to mean that the set A is a subset of the set B, that is, every element of A is also an element of B. For example, $\mathbb{N} \subseteq \mathbb{Z}$.

Example 1.1.1. Consider the set of integers $\mathbb{Z}$. We evaluate the following truth sets:

1. $\{x \in \mathbb{Z} : x \text{ is a prime number}\} = \{2, 3, 5, 7, 11, \dots\}$.
2. $\{x \in \mathbb{Z} : x \text{ is divisible by } 3\} = \{\dots, -12, -9, -6, -3, 0, 3, 6, 9, 12, \dots\}$.
3. $\{z \in \mathbb{Z} : z^2 \leq 1\} = \{-1, 0, 1\}$.
4. $\{x \in \mathbb{Z} : x^2 \leq 1\} = \{-1, 0, 1\}$.

Interval Notation

An interval is a set consisting of all the real numbers that lie between two given real numbers a and b, where $a < b$. The numbers a and b are referred to as the *endpoints* of the interval. In particular, a is called the *left endpoint* and b is called the *right endpoint*. A point in an interval that is not an endpoint is called an *interior point*. An interval may or may not include its endpoints.

1. The open interval (a, b) is defined to be $(a, b) = \{x \in \mathbb{R} : a < x < b\}$.
2. The closed interval $[a, b]$ is defined to be $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.
3. The left-closed interval $[a, b)$ is defined to be $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$.
4. The right-closed interval $[a, b)$ is defined to be $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$.

Finally, we now define the following *unbounded intervals*, where a is any real number.

1. The interval (a, ∞) is defined to be $(a, \infty) = \{x \in \mathbb{R} : a < x\}$.
2. The interval $[a, \infty)$ is defined to be $[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$.
3. The interval $(-\infty, a)$ is defined to be $(-\infty, a) = \{x \in \mathbb{R} : x < a\}$.
4. The interval $(-\infty, a]$ is defined to be $(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$.
5. The interval $(-\infty, \infty)$ is $\mathbb{R}$, the set of all real numbers.

The symbol ∞ denotes 'infinity' and it is not a number. The notation ∞ it is just a useful symbol that allows us to represent intervals that are 'without an end.' Similarly, the notation $-\infty$ is used to denote an interval 'without a beginning.'

Problem 1.1.2. Using interval notation, evaluate the following truth sets:

(1) $\{x \in \mathbb{R} : x^2 - 1 < 3\}$.

(2) $\{x \in \mathbb{R}^+ : (x-1)^2 > 1\}$.

(3) $\{x \in \mathbb{R}^- : x > \frac{1}{x}\}$.

Solution.

(1) We first solve the inequality $x^2 - 1 < 3$ for x^2 obtaining $x^2 < 4$. The solution to this latter inequality is $-2 < x < 2$. Thus, $\{x \in \mathbb{R} : x^2 - 1 < 3\} = (-2, 2)$.

(2) We are looking for all the positive real numbers x that satisfy the inequality $(x-1)^2 > 1$. We see by inspection, that the solution consists of all real numbers $x > 2$. So, $\{x \in \mathbb{R}^+ : (x-1)^2 > 1\} = (2, \infty)$.

(3) We need to find all the negative real numbers x that satisfy $x > \frac{1}{x}$. We conclude $x^2 < 1$. So, we must have $-1 < x < 0$. So, $\{x \in \mathbb{R}^- : x > \frac{1}{x}\} = (-1, 0)$.

Definition. A positive rational number $\frac{m}{n}$ is in **reduced form** if $m \in \mathbb{N}$ and $n \in \mathbb{N}$ have no common factors greater than 1.

Example. $\frac{4}{3}$ is in reduced form, $\frac{12}{9}$ is not in reduced form because 12 and 9 have a common factor greater than 1. Clearly every positive rational number can be put into reduced form.

Lemma 1.1.3. Let $a, b \in \mathbb{Z}$. If p is a prime and p divides ab, then either p divides a or p divides b.

Theorem 1.1.4. Let $p \in \mathbb{N}$ be a prime number. Then $\sqrt{p}$ is an irrational number.

Proof. Assume $p \in \mathbb{N}$ is prime. We prove that $\sqrt{p}$ is irrational. Assume, for a contradiction, that $\sqrt{p}$ is rational. Thus, (i) $\sqrt{p} = \frac{m}{n}$ for some $m, n \in \mathbb{N}$ and $n \neq 0$. We shall assume that $\frac{m}{n}$ has been put into reduced form. By squaring both sides of (i) we obtain $p = \frac{m^2}{n^2}$. Hence, we conclude that (ii) $m^2 = pn^2$. Hence, p evenly divides m^2. Since p is is a prime, p evenly divides m by Lemma 1.1.3. So, $m = pk$ for some $k \in \mathbb{N}$. After substituting $m = pk$ in (ii), we conclude $p^2k^2 = pn^2$. Therefore, $n^2 = pk^2$. Thus, p evenly divides n^2, and so, p evenly divides n. Hence, m and n have p as a common factor. It follows that $\frac{m}{n}$ is not in reduced form. Contradiction. □

1.1.2 How to Prove an Equation

Equations play a critical role in modern mathematics. In this text we will establish many theorems that will require us to know how to correctly prove an equation. Because this knowledge is so important and fundamental, our first proof strategy presents two correct methods that we shall use when proving equations.

Proof Strategy 1.1.5. To prove a new equation $\varphi = \psi$ there are two approaches:

(a) Start with one side of the equation and derive the other side.

(b) Perform operations on the given equations to derive the new equation.

We now apply strategy 1.1.5(a) to prove an well known algebraic identity.

Theorem 1.1.6. Let a and b be arbitrary real numbers. Then $(a+b)(a-b) = a^2 - b^2$.

Proof. We[2] will start with the left hand side $(a+b)(a-b)$ and derive the right hand side as follows:

$$\begin{aligned}(a+b)(a-b) &= a(a-b) + b(a-b) && \text{by the distribution property}\\ &= a^2 - ab + ba - b^2 && \text{by the distribution property}\\ &= a^2 - b^2 && \text{by algebra.}\end{aligned}$$

Thus, we have that $(a+b)(a-b) = a^2 - b^2$. □

[2] Most mathematicians use the term "we" in their proofs. This is considered polite and is intended to include the reader in the discussion.

We now apply strategy 1.1.5(b) to prove a new equation from some given equations.

Theorem 1.1.7. Let m, n, i, j be integers. Suppose that $m = 2i + 5$ and $n = 3j$. Then $mn = 6ij + 15j$.

Proof. We are given that $m = 2i + 1$ and $n = 2j$. By multiplying corresponding sides of these two equation, we obtain $mn = (2i + 5)(3j)$. Thus, by algebra, we conclude that $mn = 6ij + 15j$. □

Remark 1.1.8. To prove that an equation $\varphi = \psi$ is true, it is not a correct method of proof to *assume* the equation $\varphi = \psi$ and then work on both sides of this equation to obtain an identity.

The method described in Remark 1.1.8 is a fallacious one and if applied, can produce false equations. For example, this fallacious method can be used to derive the equation $-1 = 1$. To illustrate this, let us assume the equation $-1 = 1$. Now square both sides, obtaining $(-1)^2 = 1^2$ which results in the true equation $1 = 1$. The method cited in Remark 1.1.8 would allow us to conclude that $-1 = 1$ is a true equation. This is complete nonsense. **We never want to apply a method that can produce false equations!**

1.1.3 How to Prove an Inequality

To prove a new inequality from some given inequalities is a little more difficult than proving equations. The key difference is that you have to correctly use the Laws of Inequality.

Laws of Inequality 1.1.9. For all $a, b, c, d \in \mathbb{R}$ the following hold:

1. Exactly one of the following relations holds: $a < b$ or $a = b$ or $a > b$. (Trichotomy)
2. If $a < b$ and $b < c$, then $a < c$. (Transitivity Law)
3. If $a < b$, then $a + c < b + c$. (Adding on both sides)
4. If $a < b$ and $c > 0$, then $ac < bc$. (Multiplying by a positive)
5. If $a < b$ and $c < 0$, then $ac > bc$. (Multiplying by a negative)
6. if $a < b$ and $c < d$, then $a + c < b + d$. (Additivity)

We write $a > b$ when $b < a$, and $a \leq b$ states that $a < b$ or $a = b$. Similarly, $a \geq b$ means that $a > b$ or $a = b$. The Trichotomy Law allows us to assert that if $a \not< b$, then $a \geq b$. It should be noted that one can actually prove laws 5 and 6 from laws 1-4. Furthermore, one can also prove that $0 < 1$ and $-1 < 0$.

Problem 1.1.10. Solving inequalities.

1. Let $c < 0$. Solve $-2cx - 10 > -2c - 10$ for x.
2. Let $c < 1$. Solve $\frac{x-3}{c-1} > c + 1$ for x.

Theorem 1.1.11. Let a, b, c be a real numbers and suppose that $a < b$. Then $a - c < b - c$.

Proof. Let a, b, c be a real numbers and suppose that (1) $a < b$. From the inequality law 3 we obtain $a + (-c) < b + (-c)$. Thus, by algebra, we conclude that $a - c < b - c$. □

The following principles of inequality follow from the Laws of Inequality 1.1.9, and are frequently applied in real analysis.

Substitution Principles of Inequality 1.1.12. Let a, p, x, y real numbers. Then the following hold:

(1) Given the sum $a + x$, you can conclude that $a + x < a + y$, if $x < y$.
(Replacing a summand with a larger value yields a larger sum.)

(2) Given the sum $a + x$, you can conclude that $a + x > a + y$, if $x > y$.
(Replacing a summand with a smaller value yields a smaller sum.)

(3) Given the product px where $p > 0$, you can conclude that $px < py$, if $x < y$.
(Replacing a factor with a larger value yields a larger product.)

(4) Given the product px where $p > 0$, you can conclude that $px > py$, if $x > y$.
(Replacing a factor with a smaller value yields a smaller product.)

Principle (1) holds for $\leq$ as well (i.e., upon replacing both occurrences of $<$ in (1) with $\leq$). The above (2) also holds for $\geq$. Moreover, (3) holds for $\leq$ when $p \geq 0$; and (4) holds for $\geq$ when $p \geq 0$.

We will provide two proofs of Theorems 1.1.13–1.1.15 below. The first prove uses the Laws of Inequality 1.1.9. The second proof uses the Substitution Principles of Inequality 1.1.12.

Theorem 1.1.13. If $x \geq 3$, then $x^2 > 2x + 1$.

First Proof. Assume that $x \geq 3$. Since $x > 1$, we have that $2x + x > 2x + 1$. So $3x > 2x + 1$. As $x \geq 3$, we see that $xx \geq 3x$. So $x^2 \geq 3x$. Thus, $x^2 \geq 3x > 2x + 1$. Hence, $x^2 > 2x + 1$. □

Second Proof. Assume that $x \geq 3$. We show that $x^2 > 2x + 1$ as follows:

$$\begin{aligned} x^2 &= xx && \text{by algebra} \\ &\geq 3x && \text{as } x \geq 3 \text{ (see 1.1.12(4))} \\ &= 2x + x && \text{by algebra} \\ &> 2x + 1 && \text{because } x > 1 \text{ (see 1.1.12(1)).} \end{aligned}$$

Therefore, $x^2 > 2x + 1$. □

Theorem 1.1.14. Let a, b, c, d be a real numbers and suppose that $a < b$ and $c < d$. Then $a + c < b + d$.

First Proof. Let a, b, c, d be a real numbers satisfying (1) $a < b$ and (2) $c < d$. We prove that $a + c < b + d$. From (1) and law 3 of the Laws of Inequality 1.1.9, we obtain $a + c < b + c$. From (2) and law 3 again, we conclude that $b + c < b + d$. So, $a + c < b + c < b + d$. Therefore, $a + c < b + d$. □

Second Proof. Let a, b, c, d be a real numbers satisfying $a < b$ and $c < d$. We show that $a + c < b + d$ as follows:

$$\begin{aligned} a + c &< b + c && \text{as } a < b \text{ (see 1.1.12(1))} \\ &< b + d && \text{as } c < d \text{ (see 1.1.12(1)).} \end{aligned}$$

Therefore, $a + c < b + d$. □

Theorem 1.1.15. Suppose a and b are real numbers. If $0 < a < b$, then $a^2 < b^2$.

First Proof. Assume $0 < a < b$. We show that $a^2 < b^2$. Since $0 < a < b$, we conclude that $a < b$ and that a, b are positive numbers. Multiplying both sides of the inequality $a < b$ by the positive a gives the inequality (i) $a^2 < ab$, and multiplying both sides of the inequality $a < b$ by the positive b gives the inequality (ii) $ab < b^2$. Thus, (i) and (ii) give $a^2 < ab < b^2$. Thus, $a^2 < b^2$. Therefore, if $0 < a < b$, then $a^2 < b^2$. □

Second Proof. Assume $0 < a < b$. We show that $a^2 < b^2$ as follows:

$$\begin{aligned} a^2 &= aa && \text{by algebra} \\ &< ab && \text{as } a < b \text{ (see 1.1.12(3))} \\ &< bb && \text{as } a < b \text{ (see 1.1.12(3))} \\ &= b^2 && \text{by algebra.} \end{aligned}$$

Therefore, $a^2 < b^2$. □

Theorem 1.1.16. Suppose a and b are real numbers. If $0 < a < b$, then $\sqrt{a} < \sqrt{b}$.

Proof. Suppose $0 < a < b$. We will prove that $\sqrt{a} < \sqrt{b}$. Suppose, for a contradiction, that $\sqrt{b} \leq \sqrt{a}$. If $\sqrt{b} = \sqrt{a}$, then $b = (\sqrt{b})^2 = (\sqrt{a})^2 = a$. Contradiction. If $\sqrt{b} < \sqrt{a}$, then $b = (\sqrt{b})^2 < (\sqrt{a})^2 = a$ by Theorem 1.1.15, and so $b < a$. Contradiction. □

Theorem 1.1.17. Let a, b, c, d be positive real numbers satisfying $a < b$ and $c < d$. Then $ac < bd$.

Proof. Let a, b, c, d be positive real numbers satisfying (1) $a < b$ and (2) $c < d$. We shall prove that $ac < bd$. From (1) we conclude that $ac < bc$ because $c > 0$. From (2) we obtain $bc < bd$ because $b > 0$. So, $ac < bc < bd$. Therefore, $ac < bd$. □

Theorem 1.1.18. Suppose a, b, x, y are all positive real numbers, that is, suppose $a, b, x, y > 0$. If $a \leq b$ and $x \leq y$, then $ax \leq by$.

Proof. Suppose $a, b, x, y > 0$, $a \leq b$ and $x \leq y$. We prove that $ax \leq by$. There are several cases to consider. Suppose $a = b$ and $x = y$. Then $ax = by$ and so, $ax \leq by$. Suppose $a = b$ and $x < y$. Then $ax < ay = by$ and so, $ax \leq by$. Suppose $a < b$ and $x = y$. Then $ax < bx = by$ and so, $ax \leq by$. Suppose $a < b$ and $x < y$. Then $ax < by$ by Theorem 1.1.17. So, $ax \leq by$. □

One can also prove the following extensions of Theorem 1.1.15 and Theorem 1.1.16.

Theorem 1.1.19. If $0 < a < b$, then $a^n < b^n$ and $a^{\frac{1}{n}} < b^{\frac{1}{n}}$, for all $n \in \mathbb{N}$.

1.1.4 Important Properties of Absolute Value

Given a real number x, the absolute value of x, denoted by $|x|$, is defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

For all $a, b, x, c \in \mathbb{R}$, where $c > 0$, we have

1. $|x| < c$ if and only if $-c < x < c$
2. $|x| > c$ if and only if $x < -c$ or $x > c$
3. $|-x| = |x|$
4. $x \leq |x|$ and $-x \leq |x|$
5. $|ab| = |a||b|$
6. $|a + b| \leq |a| + |b|$ (triangle inequality)
7. $\big|\, |a| - |b| \,\big| \leq |a - b|$ (backward triangle inequality)
8. $|a| - |b| \leq |a - b|$

9. $|b| - |a| \leq |a - b|$

We will prove the following three theorems in class:

Theorem. Let $\delta > 0$. If $|x-1| < \delta$, then $|3x-3| < 3\delta$.

Theorem. Let $\delta > 0$. If $|x-1| < \delta$, then $|3x+5| < 3\delta + 8$.

Theorem. If $|x-1| < 2$, then $4 < |x+5|$.

We now present two theorems with complete proofs.

Theorem. For all $\varepsilon > 0$, if $|x-1| < \frac{\varepsilon}{3}$ then $|(3x+2)-5| < \varepsilon$.

Proof. Let $\varepsilon > 0$. Assume $(\star)$ $|x-1| < \frac{\varepsilon}{3}$. We prove that $|(3x+2)-5| < \varepsilon$ as follows:

$$\begin{aligned} |(3x+2)-5| &= |3(x-1)| && \text{by algebra} \\ &= 3\,|x-1| && \text{by property of absolute value} \\ &< 3\frac{\varepsilon}{3} && \text{by } (\star) \text{ and property of inequality} \\ &= \varepsilon && \text{by arithmetic.} \end{aligned}$$

Therefore, $|(3x+2)-5| < \varepsilon$. □

Theorem. For all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|x-1| < \delta$ then $|(3x+2)-5| < \varepsilon$.

Proof. Let $\varepsilon > 0$. Let $\delta = \dfrac{\varepsilon}{3}$. Assume $(\star)$ $|x-1| < \delta$. We prove that $|(3x+2)-5| < \varepsilon$ as follows:

$$\begin{aligned} |(3x+2)-5| &= |3(x-1)| && \text{by algebra} \\ &= 3\,|x-1| && \text{by property of absolute value} \\ &< 3\delta && \text{by } (\star) \text{ and property of inequality} \\ &= 3\frac{\varepsilon}{3} && \text{since } \delta = \frac{\varepsilon}{3} \\ &= \varepsilon && \text{by arithmetic.} \end{aligned}$$

Therefore, $|(3x+2)-5| < \varepsilon$. □

Exercises 1.1

1. Let x and y be real numbers. Prove that $(x-y)(x^2+xy+y^2) = x^3 - y^3$.

2. Let x and y be real numbers. Prove that $(x+y)(x^2-xy+y^2) = x^3 + y^3$.

3. Let x and y be real numbers. Prove that $(x+y)^2 = x^2 + 2xy + y^2$.

4. Let x and y be real numbers. Using exercise 3, prove that $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.

5. Let φ be the positive real number satisfying the equation $\varphi^2 - \varphi - 1 = 0$. Prove that $\varphi = \frac{1}{\varphi - 1}$.

6. Let φ be as in exercise 5. Let $a \neq b$ be real numbers satisfying $\frac{b}{a} = \varphi$. Prove that $\frac{a}{b-a} = \varphi$.

7. Let x be a real number such that $x > 1$. Prove that $x^2 > x$.

8. Let x be a real number where $x < 0$. Prove that $x^2 > 0$.

9. Let x be a real number where $x > 0$. Prove that $x^2 > 0$.

10. Let x be a real number where $x \neq 0$. Using exercises 8 and 9, prove that $x^2 > 0$.

11. Let a and b be real numbers where $a \neq b$. Using exercise 10 prove that $a^2 + b^2 > 2ab$.

12. Let x be a real number so that $x^2 > x$. Must we conclude that $x > 1$?

13. Let x be a real number satisfying $0 < x < 1$. Prove that $x^2 < x$.

14. Let x be a real number where $x^2 < x$. Must we conclude that $0 < x < 1$?

15. Let a and b are real numbers where $a < b$. Prove that $-a > -b$.

16. Let a, b be positive real numbers and let c, d be negative real numbers. Suppose $a < b$ and $c < d$. Prove that $ad > bc$.

17. Find a counterexample showing that the following conjecture is false: *Let a, b, c, d be whole numbers satisfying $\frac{a}{b} \leq \frac{c}{d}$. Then $a \leq c$ and $b \leq d$.*

18. Find a counterexample showing that the following conjecture is false: *Let $m \geq 0$ and $n \geq 0$ be integers. Then $m + n \leq m \cdot n$.*

19. Find a counterexample showing that the following conjecture is false: *Let $x \geq 0$ and $y \geq 0$ be real numbers. Then $\sqrt{x + y} = \sqrt{x} + \sqrt{y}$.*

20. Let a, b, c, d be real numbers. Suppose that $a + b = c + d$ and $a \leq c$. Prove that $d \leq b$. [Hint: $x \leq y$ if and only if $x - y \leq 0$.]

21. Let $a > 0$ be a real number. Prove that $\frac{1}{a} > 0$.

22. Suppose that $0 < a < b$. Prove that $\frac{1}{b} < \frac{1}{a}$.

23. Let x and y be real numbers where $x > 0$. Using Exercise 21, prove that If $xy > 0$, then $y > 0$.

24. Let $\delta > 0$. Prove that if $|x - 5| < \frac{\delta}{3}$, then $|3x - 15| < \delta$.

25. Let $\delta > 0$. Prove that if $|x - 5| < \delta$, then $|x + 3| < \delta + 8$.

26. Prove that if $|x + 5| < 1$, then $1 < |x + 3|$.

27. Let $\delta > 0$. Prove that if $|x - 3| < \delta$, then $\left|x^2 - 9\right| < \delta(\delta + 6)$.

28. Prove that for every real number $x > 3$, there exists a real number $y < 0$ such that $x = \frac{3y}{2+y}$.

29. Prove that for all real numbers x, if $x > 1$ then $0 < \frac{1}{x} < 1$.

30. Using interval notation, evaluate the following truth sets: (a) $\{x \in \mathbb{R}^+ : x > \frac{1}{x}\}$. (b) $\{x \in \mathbb{R}^- : x^2 > \frac{1}{x}\}$. (c) $\{x \in \mathbb{R}^+ : x > \frac{1}{x}$ and $x > 2\}$. (d) $\{x \in \mathbb{R}^- : x > \frac{1}{x}$ and $x \not> -\frac{1}{2}\}$.

1.2 Sets

In modern mathematics, many of the most important ideas are expressed in term of sets. A *set* is just a collection of objects. These objects are referred to as the *elements* of the set. These elements can be numbers, ordinary objects, words, other sets, functions, etc. An object a may or may not belong to a given set A. If a belongs to the set A then we say that *a is an element of A*, and we write $a \in A$. Otherwise, a is not an element of A and we write $a \notin A$.

1.2.1 Basic Definitions of Set Theory

Definition 1.2.1. The following set notation is used throughout mathematics.

1. For sets A and B, we write $A = B$ to mean that both sets have exactly the same elements.
2. For sets A and B, we write $A \subseteq B$ to assert that the set A is a subset of the set B, that is, every element of A is also an element of B.
3. For sets A and B, we write $A \subset B$ to state that A is a **proper** subset of the set B; that is, $A \subseteq B$ and $A \neq B$.
4. We write $\emptyset$ for the empty the set, that is, the set with no members.
5. If A is a finite set, then $|A|$ represents the number of elements in A.

Venn diagrams are geometric shapes that are used to depict sets and their relationships. In Figure 1.1 we present a Venn diagram which illustrates the subset relation, a very important concept in set theory and mathematics.

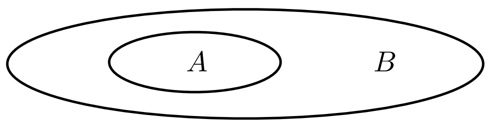
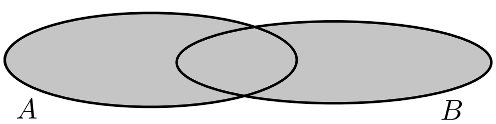


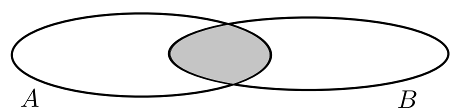
Figure 1.1: Venn diagram of $A \subseteq B$

1.2.2 Set Operations

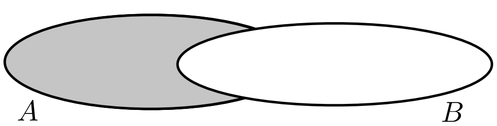
The language of set theory is used in the definitions of nearly all of mathematics. There are three important and fundamental operations on sets that we shall now discuss: the intersection, the union and the difference of two sets. We illustrate these four set operations in Figure 1.2 using Venn diagrams. Shading is used to focus one's attention on the result of each set operation.



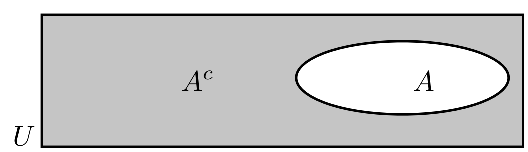
1. Venn diagram of $A \cup B$



2. Venn diagram of $A \cap B$



3. Venn diagram of $A \setminus B$



4. Venn diagram of A^c

Figure 1.2: Set Operations

Definition 1.2.2. Given sets A and B we can build new sets using the **set operations**:

1. $A \cup B = \{x : x \in A \text{ or } x \in B\}$ is the **union** of A and B.
2. $A \cap B = \{x : x \in A \text{ and } x \in B\}$ is the **intersection** of A and B.
3. $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$ is the **set difference** of A and B (also stated in English as A "minus" B).

4. Given a universe of objects U and $A \subseteq U$, the set $A^c = U \setminus A = \{x \in U : x \notin A\}$ is called the **complement** of A.

Example 1.2.3. Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8, 10, 12\}$. Then

1. $A \cup B = \{1, 2, 3, 4, 5, 6, 8, 10, 12\}$.
2. $A \cap B = \{2, 4, 6\}$.
3. $A \setminus B = \{1, 3, 5\}$, and $B \setminus A = \{8, 10, 12\}$.

Problem 1.2.4. Recalling the notation (see page 7) for intervals on the real line, evaluate the result of the following set operations:

1. $(-3, 2) \cap (1, 3)$.
2. $(-3, 4) \cup (0, \infty)$.
3. $(-3, 2) \setminus [1, 3)$.

Solution. While reading the solution to each of these items, it may be helpful to sketch the relevant intervals on the real line.

1. Since $x \in (-3, 2) \cap (1, 3)$ if and only if $x \in (-3, 2)$ and $x \in (1, 3)$, we see that x is in this intersection precisely when x satisfies both (a) $-3 < x < 2$ and (b) $1 < x < 3$. We see that the only values for x that satisfies both (a) and (b) are those such that $1 < x < 2$. Thus, $(-3, 2) \cap (1, 3) = (1, 2)$.
2. Since $x \in (-3, 4) \cup (0, \infty)$ if and only if $x \in (-3, 4)$ or $x \in (0, \infty)$, we see that x is in this union precisely when x satisfies either (a) $-3 < x < 4$ or (b) $0 < x$. We see that the only values for x that satisfies either (a) or (b) are those such that $-3 < x$. Thus, $(-3, 4) \cup (0, \infty) = (-3, \infty)$.
3. Since $x \in (-3, 2) \setminus [1, 3)$ if and only if $x \in (-3, 2)$ and $x \notin [1, 3)$, we see that x is in this set difference precisely when x satisfies (a) $-3 < x < 2$ and (b) not $(1 \le x < 3)$. We see that the only values for x that satisfies both (a) and (b) are those such that $-3 < x < 1$. Thus, $(-3, 2) \setminus [1, 3) = (-3, 1)$.

1.2.3 Indexed Families of Sets

Given a property $P(x)$ we can form the truth set $\{x : P(x)\}$ when the universe is understood. There is another way to construct sets. For example, consider the set S of all perfect squares, that is, the set of all numbers of the form n^2 for some natural number n. We can define S in two ways:

1. $S = \{x : (\exists n \in \mathbb{N})\,(x = n^2)\} = \{1, 4, 9, 16, 25, \cdots\}$.
2. $S = \{n^2 : n \in \mathbb{N}\} = \{1, 4, 9, 16, 25, \cdots\}$.

In item 1, we have expressed S as a truth set. Item 2 offers an alternative method for constructing the same set S. This alternative method is a special case of the following technique for constructing sets from the set $\mathbb{N}$ of natural numbers. Suppose for each $i \in \mathbb{N}$ we have that n_i is some object. Then we can form the set $S = \{n_i : i \in \mathbb{N}\}$ of all such objects. In this case, the set $\mathbb{N}$ is called the index set and the set S is called an indexed set or indexed family. Since this concept is used so often in mathematics, we will now formulate this idea in terms of a general definition.

Definition 1.2.5. let I be *any* set and for each $i \in I$ let x_i be some *object.* Then we can form the set $S = \{x_i : i \in I\}$. The set I is called the *index set* and the set S is called an *indexed set* or an *indexed family*

We can describe the set S in Definition 1.2.5 in two ways: As a truth set and as an indexed set, respectively:

$$S = \{x : (\exists i \in I)\,(x = x_i)\} \text{ and } S = \{x_i : i \in I\}.$$

Problem 1.2.6. Explain what the the following statements mean.

1. $y \in \{\sin(x) : x \in \mathbb{Q}\}$.
2. $\{x_i : i \in I\} \subseteq A$.
3. $\{x_i : i \in I\} \nsubseteq A$.

Solution. The first statement $y \in \{\sin(x) : x \in \mathbb{Q}\}$ means that $y = \sin(x)$ for some $x \in \mathbb{Q}$. The second statement $\{x_i : i \in I\} \subseteq A$ means that $x_i \in A$ for every $i \in I$. Finally, the third statement $\{x_i : i \in I\} \nsubseteq A$ means that $x_i \notin A$ for some $i \in I$.

Definition 1.2.7. A set $\mathcal{F}$, whose elements are sets, is called a **family of sets**.

Definition 1.2.8. Let I be *any* set and for each $i \in I$ let C_i be a *set.* Then we can form the set $\mathcal{F} = \{C_i : i \in I\}$. The set I is called the **index set** and $\mathcal{F}$ is called an **indexed family of sets**.

Example 1.2.9. Suppose for each natural number n we define the set $A_n = \{0, 1, 2, \ldots, n\}$. Then $\mathcal{F} = \{A_n : n \in \mathbb{N}\} = \{A_1, A_2, A_3, \ldots\}$ is an indexed family of sets, where the set $\mathbb{N}$ of natural numbers is the index set.

Example 1.2.10. For each real number $x > 0$, let $B_x = \{y \in \mathbb{R} : -x < y < x + 1\}$, that is, $B_x = (-x, x + 1)$. Define the indexed family of sets by $\mathcal{F} = \{B_x : x \in \mathbb{R}^+\}$, where $\mathbb{R}^+$ is the index set. Note that $B_2 \cap B_{\frac{5}{2}} = (-2, 3) \cap (-\frac{5}{2}, \frac{9}{2}) = (-2, 3)$.

Example 1.2.11. Let $I = \{i \in \mathbb{R} : i > 1\}$, the set of all real numbers greater than 1. Suppose that for each real number $i \in I$ we let $B_i = [-i, \frac{1}{i}]$, that is, $B_i = \{x \in \mathbb{R} : -i \leq x \leq \frac{1}{i}\}$. Define the indexed family of sets by $\mathcal{F} = \{B_i : i \in I\}$. Note that $B_2 \cap B_{\frac{5}{2}} = [-2, \frac{1}{2}] \cap [-\frac{5}{2}, \frac{2}{5}] = [-2, \frac{2}{5}]$.

1.2.4 Generalized Unions and Intersections

Given two sets A and B we can form the union $A \cup B$ and the intersection $A \cap B$ of these sets. In mathematics we often need to form the union and intersection of many more than just two sets. To see how this is done, we need to generalize the operations of union and intersection so that they will apply to more than just two sets. We will first extend the notions of union and intersection to a finite number of sets, and then to an infinite number of sets.

We know that $x \in A \cup B$ means that x is in at least one of the two sets A and B. This notion of union can be easily extended to more than two sets. For finitely many sets, say $A_1, A_2, \ldots A_n$, we shall say that x is in the union

$$A_1 \cup A_2 \cup \cdots \cup A_n$$

when x is in *at least one of the sets* $A_1, A_2, \ldots A_n$; that is, $x \in A_i$ for *some* $1 \leq i \leq n$. There is simpler way to denote this finite union. Using $I = \{1, 2, \ldots, n\}$ as an index set, we shall write

$$\bigcup_{i \in I} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

and so, $x \in \bigcup_{i \in I} A_i$ means that $x \in A_i$ for some $i \in I$.

We also know that $x \in A \cap B$ means that x is in both of the two sets A and B. We will extend this operation to more than two sets. For finitely many sets, say $A_1, A_2, \ldots A_n$, we shall say that x is in the intersection

$$A_1 \cap A_2 \cap \cdots \cap A_n$$

when x is in *every one one of the sets* $A_1, A_2, \ldots A_n$; that is, $x \in A_i$ for *every* $1 \leq i \leq n$. There is easier way to express this finite intersection. Using $I = \{1, 2, \ldots, n\}$ as an index set, we shall write

$$\bigcap_{i \in I} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

and so, $x \in \bigcap\limits_{i \in I} A_i$ means that $x \in A_i$ for every $i \in I$.

Similarly, we can form the union and intersection of any indexed family of sets $\{C_i : i \in I\}$, where I can be finite or infinite.

Definition 1.2.12. Let $\{C_i : i \in I\}$ be an indexed family of sets. The **union** $\bigcup\limits_{i \in I} C_i$ is the set of elements x such that $x \in C_i$ for at least one $i \in I$; that is, $\bigcup\limits_{i \in I} C_i = \{x : x \in C_i \text{ for some } i \in I\}$.

Definition 1.2.13. Let $\{C_i : i \in I\}$ be an indexed family of sets. The **intersection** $\bigcap\limits_{i \in I} C_i$ is the set of elements x such that $x \in C_i$ for every $i \in I$; that is, $\bigcap\limits_{i \in I} C_i = \{x : x \in C_i \text{ for every } i \in I\}$.

Problem 1.2.14. For each $n \in \mathbb{N}$ let C_n be the closed interval $C_n = \left[1, 1 + \frac{1}{n}\right]$. Then $\{C_n : n \in \mathbb{N}\}$ is an indexed family of sets. Evaluate the sets $\bigcap\limits_{n \in \mathbb{N}} C_n$ and $\bigcup\limits_{n \in \mathbb{N}} C_n$.

Solution. We shall evaluate the union $\bigcup\limits_{n \in \mathbb{N}} C_n$ as follows:

$$\begin{aligned} x \in \bigcup_{n \in \mathbb{N}} C_n \quad &\text{iff} \;\; x \in C_n \text{ for some } n \in \mathbb{N} && \text{by def. of } \bigcup \\ &\text{iff} \;\; x \in \left[1, 1 + \frac{1}{n}\right] \text{ for some } n \in \mathbb{N} && \text{by def. of } C_n. \end{aligned}$$

Hence, $\bigcup\limits_{n \in \mathbb{N}} C_n = [1, 2]$ because $C_1 = [1, 2]$ and $C_n \subseteq [1, 2]$ for all $n \in \mathbb{N}$. We now evaluate the intersection $\bigcap\limits_{n \in \mathbb{N}} C_n$ as follows:

$$\begin{aligned} x \in \bigcap_{n \in \mathbb{N}} C_n \quad &\text{iff} \;\; x \in C_n \text{ for every } n \in \mathbb{N} && \text{by def. of } \bigcap \\ &\text{iff} \;\; x \in \left[1, 1 + \frac{1}{n}\right] \text{ for every } n \in \mathbb{N} && \text{by def. of } C_n. \end{aligned}$$

Thus, $\bigcap\limits_{n \in \mathbb{N}} C_n = \{1\}$ because $1 + \frac{1}{n}$ gets closer and closer to 1.

Problem 1.2.15. Suppose that $\{C_i : i \in I\}$ is an indexed family of sets. Explain why the the following four statements are true.

(1) $x \in \bigcup\limits_{i \in I} C_i$ means that $x \in C_i$ for some $i \in I$.

(2) $x \notin \bigcup_{i \in I} C_i$ means that $x \notin C_i$ for every $i \in I$.

(3) $x \in \bigcap_{i \in I} C_i$ means that $x \in C_i$ for every $i \in I$.

(4) $x \notin \bigcap_{i \in I} C_i$ means that $x \notin C_i$ for some $i \in I$.

Solution. We first note that the assertion $x \notin \bigcup_{i \in I} C_i$ in (2) is the negation of that in (1). Similarly, the assertion $x \notin \bigcap_{i \in I} C_i$ in (4) is the negation of that in (3).

(1) Clearly, $x \in \bigcup_{i \in I} C_i$ means $x \in C_i$ for some $i \in I$, by Definition 1.2.12. We conclude that $x \in \bigcup_{i \in I} C_i$ iff $(\exists i \in I)(x \in C_i)$.

(2) From our solution to (1), we observed that $x \in \bigcup_{i \in I} C_i$ iff $(\exists i \in I)(x \in C_i)$. Thus,

$$x \notin \bigcup_{i \in I} C_i \quad \text{iff} \quad \neg(\exists i \in I)(x \in C_i) \quad \text{iff} \quad (\forall i \in I)(x \notin C_i).$$

So, $x \notin \bigcup_{i \in I} C_i$ means $(\forall i \in I)(x \notin C_i)$, that is, $x \notin C_i$ for every $i \in I$.

(3) From Definition 1.2.13, we see that $x \in \bigcap_{i \in I} C_i$ means $x \in C_i$ for every $i \in I$. So, $x \in \bigcap_{i \in I} C_i$ iff $(\forall i \in I)(x \in C_i)$.

(4) In our solution to (3) we noted that $x \in \bigcap_{i \in I} C_i$ iff $(\forall i \in I)(x \in C_i)$. Hence,

$$x \notin \bigcap_{i \in I} C_i \quad \text{iff} \quad \neg(\forall i \in I)(x \in C_i) \quad \text{iff} \quad (\exists i \in I)[x \notin C_i].$$

So, $x \notin \bigcap_{i \in I} C_i$ means $(\exists i \in I)[x \notin C_i]$, that is, $x \notin C_i$ for some $i \in I$.

De Morgan's Laws for Families of Sets

Theorem 1.2.16. Suppose that A is a set and that $\{B_i : i \in I\}$ is an indexed family of sets. Then

(1) $A \setminus \bigcup_{i \in I} B_i = \bigcap_{i \in I} (A \setminus B_i)$.

(2) $A \setminus \bigcap_{i \in I} B_i = \bigcup_{i \in I} (A \setminus B_i)$.

Proof. We shall prove only (1) and leave (2) as an exercise. We prove that $A \setminus \bigcup_{i \in I} B_i = \bigcap_{i \in I} (A \setminus B_i)$.

($\subseteq$). First we prove that $A \setminus \bigcup_{i \in I} B_i \subseteq \bigcap_{i \in I} (A \setminus B_i)$. To do this, let $x \in A \setminus \bigcup_{i \in I} B_i$. We must prove that $x \in \bigcap_{i \in I} (A \setminus B_i)$. We do this as follows:

$$\begin{aligned}
x \in A \setminus \bigcup_{i \in I} B_i \ &\Rightarrow^{3}\ x \in A \text{ and } x \notin \bigcup_{i \in I} B_i && \text{by the definition of } \setminus \\
&\Rightarrow\ x \in A \text{ and } x \notin B_i \text{ for every } i \in I && \text{by the definition of } \bigcup \\
&\Rightarrow\ x \in A \setminus B_i \text{ for every } i \in I && \text{by the definition of } \setminus \\
&\Rightarrow\ x \in \bigcap_{i \in I} (A \setminus B_i) && \text{by the definition of } \bigcap.
\end{aligned}$$

[2] The arrow $\Rightarrow$ is used to abbreviate the word "implies."

Therefore, $A \setminus \bigcup_{i\in I} B_i \subseteq \bigcap_{i\in I}(A \setminus B_i)$.

($\supseteq$). We now prove that $\bigcap_{i\in I}(A \setminus B_i) \subseteq A \setminus \bigcup_{i\in I} B_i$. To do this, let $x \in \bigcap_{i\in I}(A \setminus B_i)$. We must prove that $x \in A \setminus \bigcup_{i\in I} B_i$. We do this as follows:

$$
\begin{aligned}
x \in \bigcap_{i\in I}(A \setminus B_i) &\Rightarrow x \in A \setminus B_i \text{ for every } i \in I && \text{by the definition of } \bigcap \\
&\Rightarrow x \in A \text{ and } x \notin B_i \text{ for every } i \in I && \text{by the definition of } \setminus \\
&\Rightarrow x \in A \text{ and } x \notin \bigcup_{i\in I} B_i && \text{by the definition of } \bigcup \\
&\Rightarrow x \in A \setminus \bigcup_{i\in I} B_i && \text{by the definition of } \setminus.
\end{aligned}
$$

Therefore, $\bigcap_{i\in I}(A \setminus B_i) \subseteq A \setminus \bigcup_{i\in I} B_i$. Thus, the proof of (1) is complete. □

In the proof of Theorem 1.2.16, the annotations ($\subseteq$) and ($\supseteq$) are added as a courtesy to the reader. The notation ($\subseteq$) is used to make it clear to the reader that we are proving that first set is a subset of the second set. The notation ($\supseteq$) indicates that we are proving that the second set is a subset of the first set.

1.2.5 Unindexed Families of Sets

Indexed families of sets occur frequently in mathematics. Moreover, mathematicians also deal with families of sets (see Definition 1.2.7) that are not described as an indexed set. Fortunately, by a simple change in notation, every family of sets can be expressed as an indexed set. Let $\mathcal{F}$ be a family of sets. Then $\mathcal{F} = \{C_A : A \in \mathcal{F}\}$ where $\mathcal{F}$ is the index set and $C_A = A$ for each $A \in \mathcal{F}$.

Since every family $\mathcal{F}$ of sets can be expressed as an indexed family of sets, it follows that all of the operations and theorems we presented on indexed sets also apply to families of sets. When $\mathcal{F}$ is a family of sets, the **union** $\bigcup \mathcal{F}$ is the set of elements x such that $x \in C$ for some $C \in \mathcal{F}$; that is,

$$\bigcup \mathcal{F} = \{x : x \in C \text{ for some } C \in \mathcal{F}\}.$$

The **intersection** $\bigcap \mathcal{F}$ is the set of elements x such that $x \in C$ for all $C \in \mathcal{F}$; that is,

$$\bigcap \mathcal{F} = \{x : x \in C \text{ for every } C \in \mathcal{F}\}.$$

For example, let $\mathcal{F}$ be the family of sets defined by $\mathcal{F} = \{\{1,2,9\},\{2,9\},\{4,9\}\}$. Then $\bigcup \mathcal{F} = \{1,2,4,9\}$ and $\bigcap \mathcal{F} = \{9\}$.

We have the following "unindexed" version of De Morgan's Theorem 1.2.16.

Theorem 1.2.17. Suppose that A is a set and that $\mathcal{F}$ is a family of sets. Then

(1) $A \setminus \bigcup \mathcal{F} = \bigcap \{A \setminus B : B \in \mathcal{F}\}$,

(2) $A \setminus \bigcap \mathcal{F} = \bigcup \{A \setminus B : B \in \mathcal{F}\}$.

Exercises 1.2

1. Recalling our discussion on interval notation on page 7, evaluate the following set operations:

(a) $(-2, 0) \cap (-\infty, 2)$.

(b) $(-2, 4) \cup (-\infty, 2)$.

(c) $(-\infty, 0] \setminus (-\infty, 2]$.

(d) $\mathbb{R} \setminus (2, \infty)$.

(e) $(\mathbb{R} \setminus (-\infty, 2]) \cup (1, \infty)$.

2. Let $I = \{2, 3, 4, 5\}$, and for each $i \in I$ let $C_i = \{i, i + 1, i - 1, 2i\}$.

(a) For each $i \in I$, list the elements of C_i.

(b) Find $\bigcap_{i \in I} C_i$.

(c) Find $\bigcup_{i \in I} C_i$.

3. For each $n \in \mathbb{N}$ let O_n be the open interval $O_n = \left(1, 1 + \frac{1}{n}\right)$. Then $\{O_n : n \in \mathbb{N}\}$ is an indexed family of sets. Evaluate the sets: $\bigcap_{n \in \mathbb{N}} O_n$, and $\bigcup_{n \in \mathbb{N}} O_n$.

4. Let $I = \{i \in \mathbb{R} : 1 \leq i\} = [1, \infty)$ and for each $i \in I$, let $A_i = \{x \in \mathbb{R} : -\frac{1}{i} \leq x \leq 2 - \frac{1}{i}\}$. Express $\bigcup_{i \in I} A_i$ in interval notation, if possible. Express $\bigcap_{i \in I} A_i$ in interval notation, if possible.

5. Prove Theorem 1.2.16(2).

6. Prove the following theorems:

(a) **Theorem.** Let $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ be two indexed families of sets with indexed set I. Suppose $A_i \subseteq B_i$ for all $i \in I$. Then $\bigcup_{i \in I} A_i \subseteq \bigcup_{i \in I} B_i$ and $\bigcap_{i \in I} A_i \subseteq \bigcap_{i \in I} B_i$.

(b) **Theorem.** Let $\{A_i : i \in I\}$ and $\{B_j : j \in J\}$ be two indexed families of sets. Suppose there is an $i_0 \in I$ such that $A_{i_0} \subseteq B_j$ for all $j \in J$. Then $\bigcap_{i \in I} A_i \subseteq \bigcap_{j \in J} B_j$.

(c) **Theorem.** Suppose that A is a set and that $\{B_i : i \in I\}$ is an indexed family of sets. Then $A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} (A \cap B_i)$.

(d) **Theorem.** Suppose that A is a set and that $\{B_i : i \in I\}$ is an indexed family of sets. Then $A \cup \bigcap_{i \in I} B_i = \bigcap_{i \in I} (A \cup B_i)$.

(e) **Theorem.** Suppose that A is a set and that $\{B_i : i \in I\}$ is an indexed family of sets. Then $A \setminus \bigcap_{i \in I} B_i = \bigcup_{i \in I} (A \setminus B_i)$.

7. Let $\{B_x : x \in \mathbb{R}^+\}$ be the family of sets in Example 1.2.10. Evaluate $\bigcap_{x \in \mathbb{R}^+} B_x$ and $\bigcup_{x \in \mathbb{R}^+} B_x$.

8. Let $\{B_i : i \in I\}$ be the family of sets in Example 1.2.11. Evaluate $\bigcap_{i \in I} B_i$ and $\bigcup_{i \in I} B_i$.

1.3 Functions

Definition 1.3.1. We write $f : A \to B$ to mean that f is a **function** from the set A to the set B, that is, for every element $x \in A$ there is exactly one element $f(x)$ in B. The value $f(x)$ is called "f

of x," or "the image of x under f." The set A is called the **domain** of the function f and the set B is called the **co-domain** of the function f. In addition, we shall say that $x \in A$ is an *input* for the function f and that $f(x)$ is the resulting *output*. We will also say that x gets *mapped* to $f(x)$.

Remark 1.3.2. If $f\colon A \to B$ then every $x \in A$ is assigned exactly one element $f(x)$ in B. We say that f is **single-valued**. Thus, for every $x \in A$ and $z \in A$, if $x = z$ then $f(x) = f(z)$.

Definition 1.3.3. Given a function $f\colon A \to B$ the **range** of f, denoted by $\operatorname{ran}(f)$, is the set

$$\operatorname{ran}(f) = \{f(a) : a \in A\} = \{b \in B : b = f(a) \text{ for some } a \in A\}.$$

The range of a function is the set of all "output" values produced by the function.

Question. Let $h : X \to Y$ be a function. What does it mean to say that $b \in \operatorname{ran}(h)$? Answer: $b \in \operatorname{ran}(h)$ means that $b = f(x)$ for some $x \in A$.

Example 1.3.4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function in Figure 1.3 defined by the formula $f(x) = x^2 - x$. Then $\operatorname{ran}(f) = \{f(x) : x \in \mathbb{R}\} = \{x^2 - x : x \in \mathbb{R}\} = [-\frac{1}{4}, \infty)$.

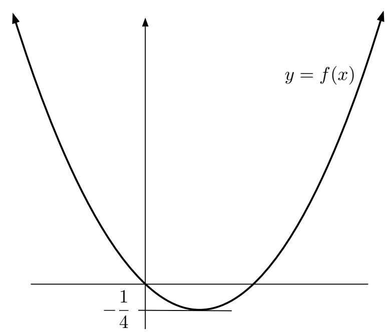


Figure 1.3: Graph of $f(x) = x^2 - x$

1.3.1 Real-Valued Functions

Real-valued functions are the focus in a calculus course and they will be the focus in this course as well. A **real-valued function** is one that has the form $f\colon D \to \mathbb{R}$, that is, the output values of the function f are real numbers. In this book, the domain of a real-valued function will typically be a set of real numbers. A **polynomial function** of **degree** n is a real-valued function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0,$$

where $a_n, \ldots, a_1, a_0$ are real number constants, n is a natural number, and $a_n \neq 0$. A **constant function** has the form $f(x) = a$ where a is a constant. A **rational function** is one that is defined as the ratio of polynomials.

1.3.2 One-To-One Functions and Onto Functions

Definition. A function $f\colon X \to Y$ is said to be **one-to-one** (or an **injection**), if distinct elements in X get mapped to distinct elements in Y; that is,

$$\text{for all } a, b \in X, \text{ if } a \neq b \text{ then } f(a) \neq f(b),$$

or equivalently,

$$\text{for all } a, b \in X, \text{ if } f(a) = f(b) \text{ then } a = b.$$

Definition. A function $f\colon X \to Y$ is said to be **onto** (or a **surjection**), if for each $y \in Y$ there is an $x \in X$ such that $f(x) = y$.

Definition. A function $f\colon X \to Y$ is said to be **one-to-one and onto** (or a **bijection**), if f is both one-to-one and onto.

1.3.3 Composition of Functions

Definition. Given two functions $f\colon X \to Y$ and $g\colon Y \to Z$, one forms the **composition function** $(g \circ f)\colon X \to Z$ by defining $(g \circ f)(x) = g(f(x))$ for all $x \in X$.

Theorem 1.3.5. If $f\colon X \to Y$ and $g\colon Y \to Z$ are one-to-one, then $(g \circ f)\colon X \to Z$ is one-to-one.

Theorem 1.3.6. If $f\colon X \to Y$ and $g\colon Y \to Z$ are onto, then $(g \circ f)\colon X \to Z$ is onto.

1.3.4 Inverse Functions

Theorem 1.3.7. Given a one-to-one function $f\colon X \to Y$, let $R = \operatorname{ran}(f)$. Then there is a function $f^{-1}\colon R \to X$ defined as follows: For each $y \in R$, $f^{-1}(y)$ is defined to be the unique element in X such that $f(x) = y$. That is, for all $y \in R$

$$f^{-1}(y) = x \quad \text{iff} \quad f(x) = y. \tag{1.1}$$

Definition. Given a one-to-one function $f\colon X \to Y$, let $R = \operatorname{ran}(f)$. The function $f^{-1}\colon R \to X$, satisfying equation (1.1) for all $y \in R$, is the **inverse function** of f.

Theorem 1.3.8. Let f be any one-to-one function $f\colon X \to Y$. Let $R = \operatorname{ran}(f)$ and let $f^{-1}\colon R \to X$ be the inverse of f. Then $(f^{-1} \circ f)\colon X \to X$ and $(f \circ f^{-1})\colon R \to R$. Moreover, the following hold:

(a) $f\colon X \to R$ is one-to-one and onto.

(b) $f^{-1}\colon R \to X$ is one-to-one and onto.

(c) $(f^{-1} \circ f)(x) = x$, for all $x \in X$.

(d) $(f \circ f^{-1})(y) = y$, for all $y \in R$.

1.3.5 Functions Acting on Sets

There are times when we are more interested in what a function does to an entire subset of its domain, rather than how it affects an individual element in the domain. Understanding this behavior on sets can allow one to better understand the function itself and can reveal some properties concerning it's domain and range. The concept of a function "acting on a set," is one that appears in every branch of mathematics.

Definition 1.3.9 (Image of a Set)**.** Let $f\colon X \to Y$ be a function. Let $S \subseteq X$. The set $f[S]$, called the **image** of S, is defined by $f[S] = \{f(x) : x \in S\} = \{y \in Y : y = f(x) \text{ for some } x \in S\}$.

Figure 1.4 illustrates Definition 1.3.9. The square S represents a subset of the domain of the function f. The image $f[S]$ is represented by a rectangle. **Note that** $f[X] = \operatorname{ran}(f)$.

Example 1.3.10. Given the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ and $S = \{-12, -3, 2, 3\}$. Then the image of S is $f[S] = \{f(x) : x \in S\} = \{|x| : x \in S\} = \{2, 3, 12\}$.

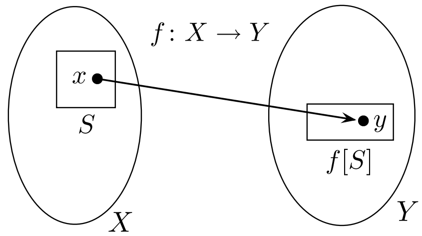


Figure 1.4: Starting with $S \subseteq X$ we construct the new set $f[S] \subseteq Y$.

Given a subset S of the domain of a function, Definition 1.3.9 allows us to construct a subset $f[S]$ of the co-domain of this function. We will now turn this process around. Our next definition will allow us to start with a subset T of the co-domain and then construct a subset of the domain.

Definition 1.3.11 (Inverse Image of a Set)**.** Let $f: X \to Y$ be a function. Let $T \subseteq Y$, that is, let T be a subset of Y. The set $f^{-1}[T]$ is the subset of X defined by $f^{-1}[T] = \{x \in X : f(x) \in T\}$. The set $f^{-1}[T]$ is called the **inverse image** of T.

A depiction of Definition 1.3.11 is given in Figure 1.5. The circle T represents a subset of the co-domain of the function f. The inverse image $f^{-1}[T]$ is represented by an ellipse.

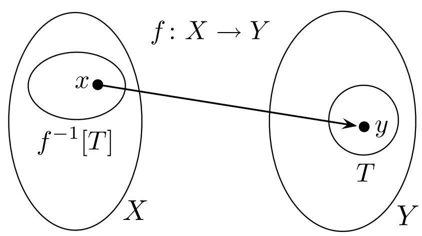


Figure 1.5: Starting with $T \subseteq Y$ we construct the new set $f^{-1}[T] \subseteq X$.

Example 1.3.12. Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ and let $T = \{-8, 2, 3\}$. Then the inverse image of T is $f^{-1}[T] = \{x \in \mathbb{R} : f(x) \in T\} = \{x \in \mathbb{R} : |x| \in T\} = \{-3, -2, 2, 3\}$.

The notation f^{-1} used in Definition 1.3.11, should not be confused with that of an inverse function. Theorem 1.3.7 implies that the inverse function exists if and only if the original function is one-to-one and onto. Definition 1.3.11 applies to all functions, even those that are not one-to-one and onto. Given a subset of the co-domain of *any* function f, the inverse image of this subset always defines a new subset of the function's domain.

The following remark states four observations that can be very useful when working with the image, or the inverse image, of a set.

Remark 1.3.13. Let $f: X \to Y$, $S \subseteq X$, $T \subseteq Y$, $a \in X$ and $b \in Y$.

1. If $a \in S$, then $f(a) \in f[S]$.
2. $b \in f[S]$ if and only if $b = f(x)$ for some $x \in S$.
3. If $a \in f^{-1}[T]$, then $f(a) \in T$.
4. If $f(a) \in T$, then $a \in f^{-1}[T]$.

Image Warning: If $f(a) \in f[S]$ then we can conclude that $f(a) = f(x)$ for some $x \in S$, by item 2 of Remark 1.3.13. Furthermore, if $f(a) \in f[S]$, we **cannot necessarily conclude** that $a \in S$. In Example 1.3.10, we have that $f(12) \in f[S]$ and yet $12 \notin S$.

Theorem 1.3.14. Let $f: X \to Y$ be a function. Let S be a subset of X, and let T be a subset of Y. Then $f[S] \subseteq T$ if and only if for all $x \in S$ we have $f(x) \in T$.

Theorem 1.3.15. Let $f: X \to Y$ be a function. Let C, D be subsets of X, and let U, V be subsets of Y. Then

(a) $f[C \cap D] \subseteq f[C] \cap f[D]$
(b) $f[C \cup D] = f[C] \cup f[D]$
(c) $f^{-1}[U \cap V] = f^{-1}[U] \cap f^{-1}[V]$
(d) $f^{-1}[U \cup V] = f^{-1}[U] \cup f^{-1}[V]$.

Proof. We shall prove only (a) and (d). Let $f: X \to Y$ be a function. Let C, D be subsets of X and let U, V be subsets of Y.

(a). We prove $f[C \cap D] \subseteq f[C] \cap f[D]$. Let $y \in f[C \cap D]$. We will prove that $y \in f[C] \cap f[D]$. Since $y \in f[C \cap D]$, there is an $x \in C \cap D$ such that $y = f(x)$ (this follows from the definition of the image of a set). Because $x \in C \cap D$, we see that $x \in C$ and $x \in D$. Therefore, $y = f(x) \in f[C]$ and $y = f(x) \in f[D]$. Thus, $y \in f[C] \cap f[D]$.

(d). We prove $f^{-1}[U \cup V] = f^{-1}[U] \cup f^{-1}[V]$.

($\subseteq$). First we prove $f^{-1}[U \cup V] \subseteq f^{-1}[U] \cup f^{-1}[V]$. Let $x \in f^{-1}[U \cup V]$. We prove $x \in f^{-1}[U] \cup f^{-1}[V]$ as follows:

$$
\begin{aligned}
x \in f^{-1}[U \cup V] &\Rightarrow f(x) \in U \cup V && \text{by definition of inverse image.}\\
&\Rightarrow f(x) \in U \text{ or } f(x) \in V && \text{by definition of } \cup.\\
&\Rightarrow x \in f^{-1}[U] \text{ or } x \in f^{-1}[V] && \text{by definition of inverse image.}\\
&\Rightarrow x \in f^{-1}[U] \cup f^{-1}[V] && \text{by definition of } \cup.
\end{aligned}
$$

Therefore, $f^{-1}[U \cup V] \subseteq f^{-1}[U] \cup f^{-1}[V]$.

($\supseteq$). Now we prove that $f^{-1}[U] \cup f^{-1}[V] \subseteq f^{-1}[U \cup V]$. Let $x \in f^{-1}[U] \cup f^{-1}[V]$. We prove that $x \in f^{-1}[U \cup V]$ as follows:

$$
\begin{aligned}
x \in f^{-1}[U] \cup f^{-1}[V] &\Rightarrow x \in f^{-1}[U] \text{ or } x \in f^{-1}[V] && \text{by definition of } \cup.\\
&\Rightarrow f(x) \in U \text{ or } f(x) \in V && \text{by def. of inverse image.}\\
&\Rightarrow f(x) \in U \cup V && \text{by definition of } \cup.\\
&\Rightarrow x \in f^{-1}[U \cup V] && \text{by def. of inverse image.}
\end{aligned}
$$

Therefore, $f^{-1}[U] \cup f^{-1}[V] \subseteq f^{-1}[U \cup V]$. This completes the proof of (d). □

Theorem 1.3.16. Let $f: X \to Y$ be a function. Let C, D be subsets of X. If f is one-to-one, then $f[C \cap D] = f[C] \cap f[D]$.

Proof. Let $f: X \to Y$ be a function. Let C, D be subsets of X. Assume that f is one-to-one. We prove that $f[C \cap D] = f[C] \cap f[D]$. By Theorem 1.3.15(a), $f[C \cap D] \subseteq f[C] \cap f[D]$. We will now show that $f[C] \cap f[D] \subseteq f[C \cap D]$. Let $y \in f[C] \cap f[D]$. We will prove that $y \in f[C \cap D]$. Since $y \in f[C] \cap f[D]$, we see that $y \in f[C]$ and $y \in f[D]$. Because $y \in f[C]$, there is an $x_1 \in C$ such that $f(x_1) = y$. Also, since $y \in f[D]$, there is an $x_2 \in D$ such that $f(x_2) = y$. Hence, $y = f(x_1) = f(x_2)$. Since f is one-to-one, we have $x_1 = x_2$. Thus, $x_1 \in D$. So, $x_1 \in C \cap D$ and therefore, $y = f(x_1) \in f[C \cap D]$. We can now conclude that $f[C \cap D] = f[C] \cap f[D]$. □

Remark 1.3.17. Suppose that $f: \mathbb{R} \to \mathbb{R}$ and let $[a, b]$ be an interval. We shall write the image of $[a, b]$ under f as $f([a, b])$ rather than $f[[a, b]]$.

Exercises 1.3

1. Using Definitions 1.3.9 and 1.3.11, explain why items 1-4 of Remark 1.3.13 are true.

2. Prove Theorem 1.3.14.

3. Prove item (b) of Theorem 1.3.15.

4. Prove item (c) of Theorem 1.3.15.

5. Given $a, b \in \mathbb{R}$ with $a > 0$, define the function $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = ax + b$. Let $U = [2, 3]$. Using interval notation, evaluate $f[U]$ and $f^{-1}[U]$.

6. Define the function $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ and let $U = [-1, 4]$. Show that

(a) $f[f^{-1}[U]] \neq U$,
(b) $f^{-1}[f[U]] \neq U$,
(c) $f[f^{-1}[U]] \neq f^{-1}[f[U]]$.

7. Let $f\colon X \to Y$ be a function and let $A \subseteq X$ and $B \subseteq X$. Prove that if $A \subseteq B$ then $f[A] \subseteq f[B]$.

8. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined in Example 1.3.10. Find $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ such that $f[A] \subseteq f[B]$ and $A \not\subseteq B$.

9. Suppose $f\colon X \to Y$ is one-to-one and let $A \subseteq X$ and $B \subseteq X$. Prove that if $f[A] \subseteq f[B]$ then $A \subseteq B$.

10. Let $f\colon X \to Y$ be a function and let $C \subseteq Y$ and $D \subseteq Y$. Prove that if $C \subset D$ then $f^{-1}[C] \subseteq f^{-1}[D]$.

11. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined in Example 1.3.12. Find $C \subseteq \mathbb{R}$ and $D \subseteq \mathbb{R}$ such that $f^{-1}[C] \subseteq f^{-1}[D]$ and $C \not\subseteq D$.

12. Suppose $f\colon X \to Y$ is onto and let $C \subseteq Y$ and $D \subseteq Y$. Prove if $f^{-1}[C] \subseteq f^{-1}[D]$ then $C \subseteq D$.

13. Let $f\colon X \to Y$ be a function. Let A be a subset of X. Prove that $A \subseteq f^{-1}[f[A]]$.

14. Suppose $f\colon X \to Y$ is one-to-one. Let $A \subseteq X$ and $x \in X$. Prove if $f(x) \in f[A]$, then $x \in A$.

15. Suppose that $f\colon X \to Y$ is one-to-one. Let $A \subseteq X$. Prove that $A = f^{-1}[f[A]]$.

16. Let $f\colon X \to Y$. Suppose $A = f^{-1}[f[A]]$ for all finite subsets A of X. Prove f is one-to-one.

17. Let $f\colon X \to Y$ be a function. Let C be a subset of Y. Prove that $f[f^{-1}[C]] \subseteq C$.

18. Assume that $f\colon X \to Y$ is an onto function. Let $C \subseteq Y$. Prove that $f[f^{-1}[C]] = C$.

19. Given $a, b \in \mathbb{R}$ with $a > 0$, define the function $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = ax + b$. Using exercises 15 and 18, prove that $f[f^{-1}[U]] = f^{-1}[f[U]]$ for every $U \subseteq \mathbb{R}$.

20. Let $f\colon X \to Y$ be a function. Let $\{C_i : i \in I\}$ be an indexed family of sets where $C_i \subseteq X$ for all $i \in I$. Prove that $f\left[\bigcup_{i \in I} C_i\right] = \bigcup_{i \in I} f[C_i]$.

Exercise Notes: Review Remark 1.3.13. Exercise 8 shows that the converse of Exercise 7 is not true for all functions. Exercise 9, however, shows that this converse is true for all one-to-one functions. Similarly, Exercise 11 shows that the converse of Exercise 10 is not true for all functions. Exercise 12 then shows that this converse is true for all functions that are onto.

1.4 Mathematical Induction

Mathematical induction is a method of proof that is frequently used to establish that certain statements are true of all the natural numbers. Before we present this method of proof, we must first discuss the well-ordering principle.

1.4.1 The Well-Ordering Principle

An important property of the set of natural numbers $\mathbb{N}$ is that any nonempty[4] subset of $\mathbb{N}$ has a least element. This property is stated as a principle that is often applied in mathematical proofs.

Well-Ordering Principle. Let S be a nonempty subset of $\mathbb{N}$. Then S has a least element.

1.4.2 Proof by Mathematical Induction

Mathematical induction is a powerful method for proving theorems about the natural numbers. Suppose you have a statement that you want to prove is true for every integer greater than or equal to the integer b. How do you prove this statement by mathematical induction? First you prove that the statement definitely holds for b. Then you have to prove that whenever the statement holds for an integer n, then it must hold for the next integer $n+1$ as well. In other words, mathematical induction is a method of proof that works by first proving the statement is true for a starting value b which is called the base step. Then one must prove the inductive step which shows that the truth of the n-th statement implies the truth of the $n+1$ statement. If the base step and the inductive step are both proven, then the statement is true for all the natural numbers $n \geq b$. We now present a proof diagram that summarizes proof by mathematical induction.

Proof Strategy 1.4.1. Let b be a fixed integer. To prove a statement of the form $(\forall n \geq b)\, P(n)$ by mathematical induction, use the diagram

Base step: Prove $P(b)$.
Inductive step: Let $n \geq b$ be arbitrary.
  Assume $P(n)$.
    Prove $P(n+1)$.

In a proof by Mathematical Induction, the proof of $P(b)$ is called the *base step* and the proof of $P(n) \to P(n+1)$ is called the *inductive step*. In the proof of $P(n) \to P(n+1)$, the statement $P(n)$ is called the **induction hypothesis** and the statement $P(n+1)$ is called the **induction conclusion**. A proof which uses the Principle of Mathematical Induction is called an **induction proof** or **proof by induction**.

In the *base step* of an induction proof, you must show that the statement $P(b)$ is true. To do so, simply replace n by b everywhere in $P(n)$ and verify that $P(b)$ holds.

The *inductive step* is more challenging. It requires you to reach the conclusion that $P(n+1)$ is true after assuming $P(n)$ is true. To prove that $P(n+1)$ is true, you should somehow try to rewrite the statement $P(n+1)$ in terms that relate to the assumption $P(n)$ (as will be illustrated in a moment) for then you will be able to make use of the assumption $P(n)$. Appealing to the assumption $P(n)$ is referred to as **using the induction hypothesis**. After establishing that $P(n+1)$ is true, the proof will be complete.

[4] A set is nonempty if it has at least one element in it.

Remark 1.4.2. The above proof strategy contains the expression "Let $n \geq b$ be arbitrary." Many mathematicians will begin such a proof with just the expression "let $n \geq b$." Such a proof is then completed under the implicit understanding that n is to be considered as an *arbitrary* real number. In this text, we will also prove statements of the form $(\forall x \in A)P(x)$ by starting the proof with the expression "let $x \in A$." The reader should then consider x to be taken as completely arbitrary.

Lemma 1.4.3. If $n \geq 4$, then $n^2 > 2n + 1$.

Proof. Assume that $(\star)$ $n \geq 4$. Since $n > 0$, we conclude that $n^2 \geq 4n = 2n + 2n$. From $(\star)$, we also see that $2n \geq 8 > 1$. Therefore, $n^2 > 2n + 1$. □

Theorem 1.4.4. For every natural number $n \geq 5$, $2^n > n^2$.

Proof. We prove, by mathematical induction, that $2^n > n^2$ for all $n \geq 5$.

Base step: For $n = 5$, we see that $2^n = 32$ and $n^2 = 25$. Thus, $2^5 > 5^2$.

Inductive step: Let $n \geq 5$ and assume the induction hypothesis that (IH) $2^n > n^2$. We show that $2^{n+1} > (n+1)^2$. Note that $2^{n+1} = 2 \cdot 2^n = 2^n + 2^n$. Hence

$$\begin{aligned} 2^{n+1} &= 2^n + 2^n && \text{by algebra} \\ &> n^2 + n^2 && \text{by the induction hypothesis (IH)} \\ &> n^2 + 2n + 1 && n^2 > 2n+1 \text{ by Lemma 1.4.3} \\ &= (n+1)^2 && \text{by factoring.} \end{aligned}$$

Hence, $2^{n+1} > (n+1)^2$ and the proof is complete. □

Theorem 1.4.5. [Sum of a geometric sequence] Let $r \neq 1$ be a fixed real number. For every integer $n \geq 0$, we have $\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}$.

Proof. Let $r \neq 1$ be a real number. We prove that $\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}$ for all $n \geq 0$, by induction.

Base step: For $n = 0$, we see that $\sum_{k=0}^{0} r^k = r^0 = 1$ and $\dfrac{r^{0+1}-1}{r-1} = 1$. Thus, $\sum_{k=0}^{0} r^k = \frac{r^{0+1}-1}{r-1}$.

Inductive step: Let $n \geq 0$ and assume the induction hypothesis that

$$\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}. \tag{IH}$$

We show that $\sum_{k=0}^{n+1} r^k = \frac{r^{(n+1)+1}-1}{r-1}$, that is, we show that $\sum_{k=0}^{n+1} r^k = \frac{r^{n+2}-1}{r-1}$ as follows:

$$\begin{aligned} \sum_{k=0}^{n+1} r^k &= \left(\sum_{k=0}^{n} r^k\right) + r^{n+1} && \text{by property of } \Sigma \text{ notation} \\ &= \frac{r^{n+1}-1}{r-1} + r^{n+1} && \text{by induction hyp (IH)} \\ &= \frac{r^{n+2}-1}{r-1} && \text{by algebra.} \end{aligned}$$

Hence, $\sum_{k=0}^{n+1} r^k = \frac{r^{n+2}-1}{r-1}$ and the proof is complete. □

Exercises 1.4

1. Prove that for every natural number $n \geq 2$, $2^n > n$.

2. Let a, b be positive real numbers where $a < b$. Prove that for every natural number $n \geq 1$, $a^n < b^n$.

3. Prove that $2^n < n!$ for all $n \geq 4$.

4. (Bernoulli's inequality) Let $x \geq -1$ be a real number. Prove that $(1+x)^n \geq 1 + nx$ for all natural numbers $n \geq 1$.

5. Suppose that $a_1, a_2, \ldots$ is a infinite sequence of real numbers satisfying $a_n \geq 0$ for all $n \geq 1$. Prove that $(1+a_1)(1+a_2)\cdots(1+a_n) \geq 1 + a_1 + a_2 + \cdots + a_n$ for all $n \geq 1$.

6. Let $c < d$ be real numbers. Suppose that $a_1, a_2, \ldots$ is a infinite sequence of real numbers satisfying $c \leq a_n \leq d$ for all $n \geq 1$. Prove that $c \leq \frac{a_1+a_2+\cdots+a_n}{n} \leq d$ for all $n \geq 1$.

7. Prove that for every natural number $n \geq 1$, we have $1 + 3 + 5 + \cdots + (2n-1) = n^2$.

8. Prove that for every natural number $n \geq 0$, $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$.

9. Prove that for every natural number $n \geq 1$, $2 + 6 + 18 + \cdots + 2 \cdot 3^{n-1} = 3^n - 1$.

10. Prove that for every natural number $n \geq 2$, $\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\cdots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$.

11. Prove that for every natural number $n \geq 1$, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

12. Prove that for every natural number $n \geq 1$, $\sum_{k=1}^{n} \frac{1}{4k^2-1} = \frac{n}{2n+1}$.

[Hint: $4(n+1)^2 - 1 = (2n+1)(2n+3)$.]

13. Prove that for every natural number $n \geq 1$, $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$.

14. Prove that for every natural number $n \geq 1$, $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Exercise Notes: For Exercise 6, multiply the induction hypothesis inequality by n and use the assumption $c \leq a_{n+1} \leq d$.