Dimensional Analysis

General Considerations

*The dimension of a variable is a monomial power function of the fundamental dimensions of the problem.*  It is a power function because the dimension appears only as a power law (not, for example, an exponential or logarithm).  It is monomial because there is only one term in the function.  In other words, you will never see some variable which has the dimensions of, say \((s^2 + s^5) \ln \text{kg}\), only dimensions like \(W \text{m}^{-2} \text{K}^{-4}\) (the Stefan-Boltzmann constant is \(\sigma = 5.67 \times 10^{-8} \text{W m}^{-2} \text{K}^{-4}\)).

Example: Newton’s Second Law

\[
\vec{F} = m \vec{a}
\]

\([\vec{a}] = \text{m s}^{-2}\)

\([m] = \text{kg}\)

\([\vec{F}] = \text{kg m s}^{-2}\)

(in the present context, \([\ ]\) is to be read “the units of” or “the dimensions of”)

Example: Stefan-Boltzmann Radiation Law

\[
F = \sigma T^4
\]

\([\sigma] = \text{W m}^{-2} \text{K}^{-4}\)

\([T] = \text{K}\)

\([F] = \text{W m}^{-2}\)

(the radiation flux, not mechanical force)

Example: Kinetic, Potential and Total Energy

\[
E = KE + PE
\]

\[
E = \frac{1}{2} m \vec{v}^2 + mgh
\]

\([\vec{v}] = \text{m s}^{-1}\)

\([m] = \text{kg}\)

\([g] = \text{m s}^{-2}\)

\([h] = \text{m}\)

\([E] = [KE] = [PE] = \text{kg m}^2 \text{s}^{-2}\)

In the Newton’s Law example, the units (dimensions) of force in the SI system must be just as they are stated, since multiplication of the mass times the acceleration must result in the appropriate multiplication of units (dimensions) as well.  In the energy example, addition or subtraction in physical (i.e. dimensional) equations must be consistent, such that each term must have the same units.
Dimensionless Products

Example: Hydrostatic Balance and Hydraulic Head

\[ h = \frac{p}{\rho g} + z_0 \]

\( p \): pressure
\( \rho \): density
\( g \): acceleration due to gravity
\( h \): hydraulic head
\( z_0 \): elevation (say, above sea level)

The left-hand side is a vertical distance (m) and the last term on the right is also a vertical distance (m). Therefore, the combination \( \frac{p}{\rho g} \) must also be a distance (m). From this fact, and the fact that \( [\rho] = \text{kg m}^{-3} \) and \( [g] = \text{m s}^{-2} \), can you find what the SI unit of pressure (the Pascal, or Pa) is in terms of kg, m and s?

If we divide the hydrostatic equation above by the quantity \( z_0 \), we obtain a dimensionless version of the same equation:

\[ h^* = p^* + 1 \]

where \( h^* = h/z_0 \) and \( p^* = p/\rho g z_0 \). The quantities \( h^* \) and \( p^* \) are called dimensionless products. There are clearly two dimensionless products in this problem. Let’s ask ourselves a question: Is there any way to determine before non-dimensionalizing the equation (that’s what we did when we intuitively divided everything by \( z_0 \)) how many dimensionless products there will be? The answer is yes.

Dimensions in the problem: \( \text{kg, m, s} \rightarrow 3 \text{ dimensions} \)
Variables in the problem: \( h, p, \rho, g, z_0 \rightarrow 5 \text{ variables} \)

\[ 5 - 3 = 2 \]

2 dimensionless products in the problem
The reason this simple method works is the realm of linear algebra, a topic beyond our scope. Suffice it to say, you can use this method any time to determine how many independent dimensionless products there will be for a given problem. Now, why are dimensionless products useful?

In our example it is clear that $h^*$ is dependent upon $p^*$. This means that once we choose the value of $p^*$, then $h^*$ is determined. So only one of these products can be chosen independently. This can be written symbolically as

$$\Pi = \phi (\Pi_1)$$

or in our case

$$h^* = \phi (p^*)$$

where $\phi$ simply expresses some functional relationship. That is, we might say that “the hydraulic head is some function of the pressure.” When we set up the problem we could find the correct relationship between variables by dimensional analysis and the correct form of the function $\phi$ by experimentation. Let’s do that.

Question: How does the depth of a uniform density fluid under the influence of gravity relate to the fluid pressure at the bottom of the fluid?

Solution: Well, we want to know about the variable $h$ in terms of $p$ and maybe the elevation of the lower boundary of the fluid in question, $z_0$. Also, we assume that the only other relevant parameters in the problem are gravity (the strength of which is measured by $g$) and the density of the fluid (measured by $\rho$). So, given $z_0$, $g$ and $\rho$, what is the functional relationship between $h$ and $p$? As outlined above, there should be 2 dimensionless products. I’m going to guess one: $h^* = h/z_0$. The other one is the one that’s more difficult to come by. I’ll do it more formally:

$$\Pi_1 = \frac{p}{z_0^a g^b \rho^c}$$

Now, since $\Pi_1$ is a dimensionless product, we can use the dimensions of $p$ to find what $a$, $b$ and $c$ are:

$$[\Pi_1] = \text{kg}^0 \text{m}^0 \text{s}^0$$

$$[p] = \text{kg}^1 \text{m}^{-1} \text{s}^{-2}$$

$$z_0^a g^b \rho^c = (\text{m})^a (\text{m} \text{s}^{-2})^b (\text{kg} \text{m}^{-3})^c = \text{kg}^c \text{m}^{a+b-3c} \text{s}^{-2b}$$

$$\left[ \frac{p}{z_0^a g^b \rho^c} \right] = \text{kg}^{1-c} \text{m}^{-1-a-b+3c} \text{s}^{-2+2b}$$

Using the first and last equations above, we find that

$$1 - c = 0$$

$$-1 - a - b + 3c = 0$$

$$-2 + 2b = 0$$

which has the solution: $a = 1$, $b = 1$, $c = 1$ so the dimensionless product $\Pi_1 = \frac{p}{z_0 g \rho}$. We have discovered, with only the aid of units, that

$$\frac{h}{z_0} = \phi \left( \frac{p}{z_0 g \rho} \right)$$

Now, we still don’t know exactly what this function $\phi$ looks like. We need to perform some experiments to determine the nature of this function. I bet we could devise a series of experiments, varying $h$ and $z_0$ (by varying the experimental set-up) as well as $\rho$ (by using different liquids); maybe not $g$. From these experiments, we could measure the pressure, $p$ and the plot $\Pi = \frac{h}{z_0}$ vs. $\Pi_1 = \frac{p}{z_0 g \rho}$. Since I know it is a linear function, I bet you could find the slope as well as the intercept, from a linear regression on the data you measured. Again, since I know the solution ($h^* = p^* + 1$), I know you should get a slope of 1 and an intercept of 1.
Exploring Alternative Energy with Dimensional Analysis

Example: Wind and Solar Power in New York State
Question: How much electrical power can a wind turbine generate?
Solution:
First we’ll solve the problem the old fashioned way, with straight physics; then we’ll do it with dimensional analysis alone. Let’s make some assumptions:

1. we know the size of the wind turbine, i.e. how long the blades are
2. we know the near-surface atmospheric density, \( \rho \) and the near-surface wind speed, \( U \)
3. we know how efficiently the wind turbine itself can turn wind kinetic energy into kinetic energy of spinning wind turbine blades (a non-dimensional fraction, \( \epsilon \))
4. we know how efficiently the turbine can convert the kinetic energy of the spinning blades into electrical energy available to the grid (a non-dimensional fraction, \( \eta \))

![Diagram for use in the wind turbine problem. On the right, note that not all of the kinetic energy of the wind is used to spin up the blades. Some of the wind kinetic energy remains downstream of the turbine. Hence the flow is weaker but not stagnant downstream of the blades. The efficiency of this process is measured by \( \epsilon \). Finally, as the blades spin, they drive a turbine which converts mechanical energy into electrical energy. This process is also not perfectly efficient; the turbine efficiency is measured by \( \eta \).]

Now, the formula for kinetic energy of a fluid, per unit volume is

\[
KE = \frac{1}{2} \rho U^2
\]
If we can find out what volume of fluid the wind turbine blades intercept every second then we can multiply the above formula by that volume and get “the kinetic energy intercepted by the blades every second” or “the power intercepted by the blades”. Multiplying this power by $\epsilon$ gives the power of the spinning blades themselves. Finally, multiplying by $\eta$ gives the power that can be delivered to the grid. Now what is the volume of air intercepted by the spinning blades every second?

![Diagram of the volume of fluid intercepted by a cross-sectional area perpendicular to the flow.](image)

**Figure 3:** Diagram of the volume of fluid intercepted by a cross-sectional area perpendicular to the flow.

If a parcel of air is at the blades at time $t$, then $\delta t$ seconds later it will have travelled $U\delta t$ meters downstream of the turbine. Since the blades trace out a circular area of $\pi R^2$, where $R$ is the length of the blades, we can calculate the volume of air intercepted by the blades in a time $\delta t$ by the formula for a cylinder (see diagram in Figure 3):

$$\delta V = \pi R^2 \delta L = \pi R^2 U \delta t$$

So the rate at which a volume of air flows past the blades is

$$\frac{\delta V}{\delta t} = \pi R^2 U$$

Now, multiplying this volume flow rate by the kinetic energy per unit volume, we arrive at a power, $P_0$, the power of the wind intercepting the blades

$$P_0 = \frac{\pi}{2} \rho R^2 U^3$$

and then of the spinning blades themselves

$$P_1 = \frac{\pi}{2} \epsilon \rho R^2 U^3$$

and, finally, the power available to the grid

$$P_2 = \frac{\pi}{2} \eta \epsilon \rho R^2 U^3$$
This means several things. If we double the size of the blades we might expect to quadruple the power output. If we double the wind speed (find a place that’s twice as windy to build it) we increase the power output by a factor of 8. Finally, if we double any of \( \rho, \epsilon \) or \( \eta \) we only double the deliverable power. Now let’s see if we can get the same scaling relationships with dimensional analysis.

The 6 relevant parameters are

\[ U, \rho, R, \epsilon, \eta, P \]

\( P \) is the power delivered to the grid. The units of power are energy per time or \([ P ] = \text{kg m}^2 \text{s}^{-3}\). The other units are \([ \rho ] = \text{kg m}^{-3}, [ U ] = \text{m s}^{-1}\), and \([ R ] = \text{m}\). So the unit system for this problem contains 3 dimensions

- \( \text{kg, m, s} \)

So then

\[ 6 - 3 = 3 \]

3 dimensionless products

The first two are obvious, \( \eta \) and \( \epsilon \). Now we can write the third dimensionless product as a function of the other two:

\[ \Pi = \phi (\Pi_0, \Pi_1) \]

or

\[ \Pi = \phi (\epsilon, \eta) \]

Now what is \( \Pi \)? By definition it is unitless, so \([ \Pi ] = \text{kg}^0 \text{m}^0 \text{s}^0\). Also, since we’re after the power \( P \), let’s lump \( P \) into \( \Pi \) and find out what combination of the other variables we need to make it dimensionless

\[ \Pi = \frac{P}{\rho^a R^b U^c} \]

So, plugging in the relevant units, we have

\[ \left[ \frac{P}{\rho^a R^b U^c} \right] = \text{kg}^{1-a} \text{m}^{2+3a-b-c} \text{s}^{-3+c} \]

and finally (since \( \Pi \) is dimensionless)

\[ 1 - a = 0 \]
\[ 2 + 3a - b - c = 0 \]
\[ -3 + c = 0 \]

These equations have the solution \( a = 1, b = 2, c = 3 \) so that

\[ \frac{P}{\rho R^2 U^3} = \phi (\epsilon, \eta) \]

and, just as before with the physics, we have

\[ P = \phi (\epsilon, \eta) \rho R^2 U^3 \]

We know how the function \( \phi \) should behave from our idea of what we mean by “efficiency”; it should be linear in \( \epsilon \) and \( \eta \). Hence, we get the same answer as before, without the factor of \( \frac{\pi}{2} \)

\[ P = \epsilon \eta \rho R^2 U^3 \]

However, since \( \pi/2 \) isn’t much different than 1, this is pretty good. Furthermore, all the scaling relationships are the same. That is, if you double the density of the fluid, you double the power; if you double the size of the blades, you quadruple the power; if you double the wind speed, you increase the power by a factor of 8.
Plugging in some numbers, let’s use $U = 5 \text{ m s}^{-1}$, $\rho = 1.22 \text{ kg m}^{-3}$, $R = 10 \text{ m}$. For the efficiencies, take a look at some info pulled off the web (source: [http://www.reuk.co.uk/Betz-Limit.htm](http://www.reuk.co.uk/Betz-Limit.htm)):

Albert Betz was a German physicist who in 1919 concluded that no wind turbine can convert more than $\frac{16}{27}$ (59.3%) of the kinetic energy of the wind into mechanical energy turning a rotor. To this day this is known as the Betz Limit or Betz’s Law. This limit has nothing to do with inefficiencies in the generator, but in the very nature of wind turbines themselves.

Wind turbines extract energy by slowing down the wind. For a wind turbine to be 100% efficient it would need to stop 100% of the wind - but then the rotor would have to be a solid disk and it would not turn and no kinetic energy would be converted. On the other extreme, if you had a wind turbine with just one rotor blade, most of the wind passing through the area swept by the turbine blade would miss the blade completely and so the kinetic energy would be kept by the wind.

![Image of wind turbine efficiencies]

**Real World Wind Turbine Power Efficiencies**

The theoretical maximum power efficiency of any design of wind turbine is 0.59 (i.e. no more than 59% of the energy carried by the wind can be extracted by a wind turbine). Once you also factor in the engineering requirements of a wind turbine - strength and durability in particular - the real world limit is well below the Betz Limit with values of 0.35-0.45 common even in the best designed wind turbines. By the time you take into account other inefficiencies in a complete wind turbine system - e.g. the generator, bearings, power transmission and so on - only 10-30% of the power of the wind is ever actually converted into usable electricity. (see the graphic above from the Iowa Energy Center, USA.)

So let’s use $\epsilon = 0.4$ and $\eta = 0.6$.

$U = 5 \text{ m s}^{-1}$, $\rho = 1.22 \text{ kg m}^{-3}$, $R = 10 \text{ m}$

$\epsilon = 0.4$, $\eta = 0.6$

This gives

$P \approx 3660 \text{ W or } 3.66 \text{ kW}$

Next, we’ll use this estimate to explore some ideas about alternative energy.
New York State Energy Consumption
What is the total power output for fossil fuel burning power stations in New York State?

Charles Poletti Power Project
Location: New York City, on the East River
Net Dependable Capability: 885,000 kilowatts
First Commercial Power: March 1977

Richard M. Flynn Power Plant
Location: Holtsville, Suffolk County
Net Dependable Capability: 135,600 kilowatts
First Commercial Power: May 1994

PowerNow! Small Power Plants
Locations: New York City (6); Brentwood, Long Island (1)
Net Dependable Capacity: 450,000 kilowatts
First Commercial Power: Summer 2001

500 MW Combined-Cycle Power Plant
Location: New York City, on the East River
Net Dependable Capability: 500,000 kilowatts
First Commercial Power: December 2005

(the information above can be found at http://www.nypa.gov/facilities/default.htm)

In order to produce enough power to replace the existing fossil fuel burning power plants in New York, we need to generate

\[
\text{Total Power} = (885 + 135.6 + 450 + 500) \times 10^3 \text{ kW} = 1970.6 \times 10^3 \text{ kW}
\]

From a previous calculation, one turbine can be estimated to generate about 3.66kW of power. Then, to eliminate the fossil fuel burning plants, we need

\[
\frac{1.9706 \times 10^6 \text{ kW}}{3.66 \text{ kW per Wind Turbine}} = 538,415 \text{ Wind Turbines}
\]

Assuming each turbine to require about 1/2 acre of land to be functional, this would require about 250 thousand acres (2.5 \times 10^5 acres). Since 1 acre = 4046.85642 m^2, then we would need to use about 1 \times 10^9 m^2 of land area. Using Figure 4, and the formula for surface area on a sphere

\[
\delta A = R^2 \cos \phi \delta \lambda \delta \phi
\]

we can estimate the total area of New York State. The figure is broken up into several 1° × 1° boxes. The radius of the Earth is \( R = 6.37 \times 10^6 \text{ m} \). First, what is the area of one of these 1° × 1° boxes?

\[
\delta A_{1^\circ \times 1^\circ} \approx R^2 \cos (43^\circ) \times 1^\circ \times 1^\circ \times \left( \frac{\pi}{180^\circ} \right)^2
\]

\[
\delta A_{1^\circ \times 1^\circ} \approx 9 \times 10^9 \text{ m}^2
\]

The two factors of \( \pi/180^\circ \) is to convert from degrees to radians in \( \delta \lambda \) and \( \delta \phi \). Then, by gross estimation, there appear to be about 14 of these boxes filled with pink, so

\[
\delta A_{\text{New York}} \approx 1.26 \times 10^{11} \text{ m}^2
\]
So the fraction of New York State land that would have a wind turbine on it is

\[
\text{Land Fraction} \approx \frac{1 \times 10^9}{1.26 \times 10^{11}} \approx 0.0079 \text{ or about } 0.79\% 
\]

Now, what if we wanted to power NYS with solar energy? Assume that the average insolation (INcoming SOLar radiATION) over one year in New York State is

\[
F \approx \tau (1 - \alpha) \cos(\phi) S_0 = 122 \text{ Wm}^{-2}
\]

\(\tau\): atmospheric transmittivity, or fraction of TOA insolation reaching surface; we have assumed \(\tau = 0.7\).
\(\phi\): Latitude of NYS; we have assumed \(\phi = 43^\circ\).
\(\alpha\): Net shortwave planetary albedo; we have assumed \(\alpha = 0.7\).

Some NASA observational data is plotted in Figure 5. The net incoming solar radiation incident upon a horizontal surface averaged over the annual cycle for a 22 year period is plotted in units of kWh per day. According to the data, the number is supposed to be something like

\[
\frac{3.5 \text{ kWh day}^{-1} \text{m}^{-2}}{24 \text{ h day}^{-1}} = 0.145 \text{ kW m}^{-2} = 145 \text{ W m}^{-2}
\]

We weren’t too far off, but in the following calculations let’s use the cited observational data. However, before going further, take a look at the plot again. Why do you think a simple correspondence with latitude
Figure 5: Net incoming solar radiation at the surface. The averages are taken over the annual cycle over the 22 year period from 1983 to 2005. The units of the observed solar flux are kWh m$^{-2}$day$^{-1}$. The data themselves can be obtained for free with registration at http://eosweb.larc.nasa.gov/sse/.

(i.e. the simple $\cos \phi$ term above) doesn’t quite work? That is, what processes give rise to the geographic (in particular, longitudinal) variability in the data?

Now, the maximum efficiency of commercially available solar panels is about 19% or 0.19. So how many square meters (and what percentage of the state) would we have to cover to eliminate fossil fuel power production in NYS? To answer this, first calculate the net power produced by solar panels (photo-voltaic cells, or PVCs) per square meter:

$$\text{power per square meter from PVCs} \approx 0.19 \times 145 \text{ W m}^{-2} \approx 27 \text{ W m}^{-2}$$

and so the area required to eliminate fossil fuel burning plants is

$$\text{Land Area} = \frac{1.9706 \times 10^9 \text{ W}}{27 \text{ W m}^{-2}} \approx 7.3 \times 10^7 \text{ m}^2$$

Since the total area of the state is about $1.26 \times 10^{11}$ m$^2$, this means that about 0.058% of the State of New York’s land area would be covered with the PVCs. Both the land fractions we calculated were less than 1%. What are the absolute numbers comparable to? I mean, are the numbers comparable to the size of a city park, Buffalo, Erie County? To give you some reference, Delaware Park is about 350 acres.