%\documentclass[12pt,twoside,draft]{article}

\documentclass[11pt,twoside,notitlepage]{article}

\usepackage{amsmath}

\usepackage{amsthm}

\usepackage{amssymb}

\usepackage{latexsym}

\usepackage[mathscr]{eucal}

\usepackage{ifthen}

%\usepackage[draft]{graphicx}

\usepackage{graphicx}

\date{}

%\usepackage[labelformat=empty]{subfig} %useful for mult-figures in a figure - defines \subfloat

%

\usepackage[labelformat=empty]{subfig}

%------------------------------- New theorems

% These will be typeset in Roman

\theoremstyle{definition}

\newtheorem{theorem}{Theorem}

\newtheorem{corollary}[theorem]{Corollary}

\newtheorem{proposition}[theorem]{Proposition}

\newtheorem{lemma}[theorem]{Lemma}

%\newtheorem{exmp}[theorem]{Example}

\newtheorem{thm}{Theorem}[theorem]

\newtheorem\*{thmA}{Theorem A}

\newtheorem\*{thmC}{Theorem \ref{thm30.7}}

\newtheorem\*{lemA}{Theorem \ref{exmp2.2}}

\newtheorem\*{lemB}{Theorem \ref{lem30.3.3}}

\newtheorem{define}{Definition}[theorem]

\newtheorem{prop}{Proposition}[theorem]

\newtheorem{lem}{Lemma}[theorem]

%\newtheorem{prob}{Problem}[theorem]

\newtheorem{cor}{Corollary}[theorem]

%\newtheorem{diagram}[theorem]{Proof Diagram}

\newtheorem{comment}[theorem]{Comment}

\newtheorem{prob}{Problem}

\newtheorem\*{rmk}{Comment}

\newtheorem\*{them}{Theorem}

\newtheorem\*{lame}{Lemma}

\newtheorem\*{thems}{Theorem\*}

\newtheorem\*{prbm}{Problem}

% These will be typeset in Roman

\theoremstyle{definition}

\newtheorem{definitions}{Definition}

\newtheorem{lemmas}[definitions]{Lemma}

\newtheorem{definition}[theorem]{Definition}

\newtheorem{exmple}[theorem]{Example}

\newtheorem{fact}[theorem]{Fact}

\newtheorem{conjecture}[theorem]{Conjecture}

\newtheorem{remark}[theorem]{Remark}

\newtheorem{claim}{Claim}

\newtheorem\*{deftn}{Definition}

\newtheorem\*{defn}{Definition}

\newtheorem\*{exercise}{Exercise}

\newtheorem\*{exple}{Example}

\newtheorem\*{solution}{\it Solution}

\newtheorem\*{analysis}{\it Proof Analysis}

\newtheorem\*{scratch}{\it Scratch Work}

\newtheorem\*{sketch}{\it Proof Diagram}

\newtheorem\*{notation}{Notation and Terminology}

\newtheorem\*{rmrk}{Remark}

\newtheorem\*{field}{$\R$ is an Ordered Field}

\newtheorem\*{note}{Attention}

\theoremstyle{definition}

\newtheorem{example}{Example}

\newtheorem{problem}{Problem}

\theoremstyle{definition}

\newtheorem\*{axiom}{Completeness Axiom}

\newtheorem\*{wop}{Well-Ordering Principle}

\newtheorem\*{indp}{Principle of Mathematical Induction}

\newtheorem{lawineq}[theorem]{Laws of Inequality}

\newtheorem{princineq}[theorem]{Useful Principles of Inequality}

\newtheorem\*{question}{Question}

\theoremstyle{definition}

\newtheorem\*{clam}{Claim}

\newtheorem{strategy}[theorem]{\bf Proof Strategy}

\newtheorem{assumption}[theorem]{\bf Assumption Strategy}

\newtheorem{diag}[theorem]{Proof Diagram}

%\numberwithin{equation}{section}

%--------------------My Definitions

\def\fnctbscale{.45}

\def\realascale{.6}

\def\realbscale{.5}

\def\altsscale{.7}

\def\x{\mathbf{x}}

\def\y{\mathbf{y}}

\def\u{\mathbf{u}}

\def\v{\mathbf{v}}

\renewcommand{\o}{\text{o}}

\def\u{\underline}

\def\fixit{{\text{$\gamma\,\,$}}}

\newcommand{\be}{\begin{enumerate}}

\newcommand{\ee}{\end{enumerate}}

\newcommand{\bi}{\begin{itemize}}

\newcommand{\ei}{\end{itemize}}

\newcommand{\bR}{\boldsymbol{\mathbb{R}} }

\newcommand{\R}{\ensuremath{\mathbb{R}} }

\newcommand{\N}{\ensuremath{\mathbb{N}} }

\def\Q{{\text{$\mathbb{Q}$} }}

\def\I{{\text{$\mathbb{I}$} }}

\def\Z{{\text{$\mathbb{Z}$} }}

\newcommand{\qqed}{\renewcommand{\qedsymbol}{}\end{proof}\vspace{-.4in}}

\newcommand{\qqqed}{\renewcommand{\qedsymbol}{}\end{proof}\vspace{-.35in}}

\newcommand{\qsed}{\renewcommand{\qedsymbol}{}\end{proof}\vspace{-.25in}}

\newcommand{\qeed}{\renewcommand{\qedsymbol}{}\end{proof}\vspace{-.8in}}

\newcommand{\ef}{\end{figure}}

\newcommand{\dab}{\quad\ }

\newcommand{\dual}{\breve }

\newcommand\zero{\boldsymbol{\theta}}

\def\iffarrow{\leftrightarrow}

\def\dom{\hbox{dom}}

\def\ran{\hbox{ran}}

\def\card{\hbox{card}}

\def\intr{\hbox{int}}

\def\bd{\hbox{bd}}

\def\cl{\hbox{cl}}

\def\l{[\![}

\def\r{]\!]}

\def\P{\mathcal{P}}

\def\F{\mathcal{F}}

\def\G{\mathcal{G}}

\def\K{\mathcal{K}}

\def\O{\mathcal{O}}

\def\C{\mathcal{C}}

\newcommand{\A}{\ensuremath{\mathcal{A}}}

\newcommand{\B}{\ensuremath{\mathcal{B}}}

\renewcommand{\iff}{\text{\,\ {\textup{ iff}}\,\ }}

\newcommand{\abs}[1]{\left\lvert#1\right\rvert}

\newcommand{\seq}[1]{\left\langle#1\right\rangle}

\newcommand{\maax}[1]{\textup{max}\!\left\{#1\right\}}

\newcommand{\miin}[1]{\textup{min}\!\left\{#1\right\}}

\newcommand{\set}[1]{\left\{#1\right\}}

\newcommand{\lb}{\left[}

\newcommand{\rb}{\right]}

\newcommand{\eps}{\varepsilon}

\newcommand{\ds}{\displaystyle}

\newcommand{\separate}{\medskip}

\newcommand{\bseparate}{\bigskip}

\newcommand{\sseparate}{\smallskip}

\newcommand{\rank}{\textup{rank}}

\newcommand{\df}[1]{{\bf #1}} % highlight the key word in a definition

\newcommand{\hspc}{\hspace{.35\textwidth}} %proof diagrams

\newcommand{\hspcc}{\hspace{.085\textwidth}}

\newcommand{\hspcntr}{\hspace{.30\textwidth}} %proof by contradiction

\newcommand{\hscc}{\hspace{2em}} %existence and uniqueness diagrams

\newcommand{\hspec}{\hspace{.035\textwidth}} %existence and uniqueness diagrams

\newcommand{\hsind}{\hspace{.05\textwidth}} %induction diagrams

\newcommand{\hspind}{\hspace{.035\textwidth}} %induction diagrams

\newcommand{\hstind}{\hspace{.01\textwidth}} %induction strong diagrams

\newcommand{\hspstind}{\hspace{.025\textwidth}} %induction strong diagrams

\renewcommand{\emptyset}{\varnothing}

%work around: used to get index entry with no cross reference attached; for example, use \iw{\protect{universal proofs|see{for all}}}

\newcommand{\see}[2]{{\it see\/} {#1}}

\newcommand{\iw}[1]{\index{iwords}{#1}} %index word

\newcommand{\is}[1]{\index{isymbols}{\thepage @#1}} %index symbol

\newlength{\bozo}

\setlength{\bozo}{.65\textwidth}

\newlength{\bozoo}

\setlength{\bozoo}{.85\textwidth}

\makeatletter

\newcommand{\landfill}{\leavevmode\leaders\hrule\@height.25ex\@depth0em\hfill\kern\z@}

\newcommand{\landfillend}{\leavevmode\leaders\hrule\@height.3ex\@depth0em\hfill\kern\z@}

\makeatother

\newcommand{\hwexercises}[1]{\subsection\*{Exercises \arabic{chapter}.\arabic{section} \rule{\bozo}{0.80ex}\landfill}{\vspace{1.00ex}} %space after between Exercise statement and `first instruction for exercises'

{#1}{\vspace{-0.5ex} %space after last exercise and `end line'

\noindent \landfillend\rule{\bozoo}{.85ex}}\vspace{.5ex}}

\newcommand{\txx}[1]{\quad\text{#1}}

\newcommand{\txxx}[1]{\hspace{2.5em}\text{#1}}

\newcommand{\txxs}[1]{\hspace{.5em}\text{#1}}

\newcommand{\hwexerciseswn}[1]{\subsection\*{Exercises \arabic{chapter}.\arabic{section} \protect\rule{\bozo}{0.85ex}\landfill}{\vspace{1.00ex}}#1{\vspace{1.5ex} %space after ex notes and `end line'

\noindent\landfill\rule{\bozoo}{0.95ex}}\vspace{.5ex}}

\newcommand{\hwex}[1]{\subsection\*{Exercises \arabic{chapter}.\arabic{section}.\arabic{subsection} \rule{\bozo}{0.85ex}\landfill}{\vspace{1.00ex}} %space after between Exercise statement and `first instruction for exercises'

{#1}{\vspace{0.0ex} %space after last exercise and `end line'

\noindent \landfillend\rule{\bozoo}{.95ex}}\vspace{.5ex}}

% Alternative Exercise List--numbered by chapter and section

\newcounter{excntr}

\newcommand{\bh}{\begin{list}{\bf\arabic{excise}.}{\usecounter{excise}\setlength{\leftmargin}{2.0em}}\setlength{\labelwidth}{4ex}\setlength{\itemsep}{.05\itemsep}} % beginning of exercise list

\newcommand{\eh}{\end{list}} % end of exercise list

% sublist

\newcounter{sbex}

\newcommand{\bhs}{\begin{list}{(\alph{sbex})}{\usecounter{sbex}\setlength{\itemsep}{.05\itemsep}\setlength{\topsep}{.5\topsep}}} % beginning of exercise sublist

\newcommand{\ehs}{\end{list}} % end of exercise sublist

\newcounter{ssbex}

\newcommand{\bhss}{\begin{list}{(\roman{ssbex})}{\usecounter{ssbex}\setlength{\itemsep}{.05\itemsep}\setlength{\topsep}{.5\topsep}}} % beginning of exercise subsublist

\newcommand{\ehss}{\end{list}} % end of exercise subsublist

%

\newcounter{excise}

\newcommand{\bx}{\begin{list}{(\arabic{chapter}.\arabic{section}.\arabic{excise})}{\usecounter{excise}}} % beginning of exercise list

\newcommand{\ex}{\end{list}} % end of exercise list

\newcommand{\bxx}{\begin{list}{(\alph{prob})}{\usecounter{prob}}} %subitem exercise list

\newcommand{\exx}{\end{list}}

%end of subitem exercise list

% \item\label{exnum}--assign a label to an exercise

% See Exercise \ref{c1s1}.\ref{exnum}-- to refer to this exercise where c1s1 is section label

%

% End of Alternative Exercise List

%

% Alternative List-

\newcounter{listcntr}

\newcommand{\bl}{\begin{list}{\arabic{listcntr}.}{\usecounter{listcntr}\setlength{\leftmargin}{1.5em}}\setlength{\labelwidth}{4ex}\setlength{\itemsep}{.05\itemsep}}

% makes the caption text small and moves the caption up closer to the figure

\newcommand{\captionfonts}{\small}

\makeatletter % Allow the use of @ in command names

\long\def\@makecaption#1#2{%

\vspace{.75ex}% \vskip\abovecaptionskip % can replace this line with \vspace{2ex}

\sbox\@tempboxa{{\captionfonts #1: #2}}

\ifdim \wd\@tempboxa >\hsize

{\captionfonts #1: #2\par}

\else

\hbox to\hsize{\hfil\box\@tempboxa\hfil}%

\fi

\vskip\belowcaptionskip}

\makeatother % Cancel the effect of \makeatletter

\newcommand{\el}{\end{list}} % end of alternate list

%\renewcommand{\baselinestretch}{1.27}

%\addtolength{\parindent}{.5\parindent}

\setlength{\textwidth}{6.5in}

\newlength{\mymargin}

\setlength{\mymargin}{-0.0in} %(8.5in - \textwidth)/2 - 1in

\setlength{\oddsidemargin}{\mymargin}

\setlength{\evensidemargin}{\mymargin}

\setlength{\textheight}{9.75in}

\setlength{\topmargin}{-1.05in} %(10.0in - \textheight)/2 - 1in}

%end-Set Size of paper

\newlength{\myitemsep}

%\setlength{\myitemsep}{\itemsep}

\setlength{\myitemsep}{.5\itemsep}

\newlength{\smitemsep}

\setlength{\smitemsep}{-.3\itemsep}

\newcommand{\bes}{\begin{enumerate}\itemsep=\smitemsep}

\newcommand{\ees}{\end{enumerate}}

\newcommand{\bis}{\begin{itemize}\itemsep=\myitemsep}

\newcommand{\eis}{\end{itemize}}

\newcommand{\biis}{\begin{itemize}\itemsep= \smitemsep}

\newcommand{\eiis}{\end{itemize}}

\newcommand{\txt}[1]{\quad\text{#1}}

%End of My Definitions

%\author{Instructor: Daniel Cunningham}

\date{}

\newlabel{thm29.9}{{6.1.17}{105}}

\begin{document}

\phantom{p}

\vspace{-.5in}

\centerline{\bf\Large MAT 491--Spring 2019}

\centerline{\bf\Large Final Problems in Real Analysis \& Abstract Algebra}

\medskip

\section{Real Analysis Problems}

\begin{definition}\label{defpresconv} We shall say that $f\colon D\to\R$ {\bf preserves convergent sequences}

if for all convergent sequences $\seq{x\_n}$ with $x\_n\in D$ for all $n\ge 1$, we have that $\seq{f(x\_n)}$ also converges.

\end{definition}

\begin{definition}[Partition of an Interval]\label{def29.1} Let $[c,d]$ be an interval.

\bi

\item A finite set $P=\{x\_i : 0\le i\le n\}=\set{x\_0, x\_1, x\_2, \dots, x\_n}$ of points in $[a,b]$ is called a {\bf partition} of $[a,b]$ provided that $c=x\_0<x\_1<x\_2<\cdots<x\_n=d$.

\item If $P$ and $P^\*$ are two partitions of $[c,d]$ with $P\subseteq P^\*$, then $P^\*$ is called a {\bf refinement} of $P$.

\ei

\end{definition}

\begin{rmrk} Given two partitions $P$ and $Q$ of $[c,d]$, it follows that $P\cup Q$ is a partition of $[c,d]$ and so, $P\cup Q$ is a refinement of both $P$ and $Q$.

\end{rmrk}

\begin{definition}\label{PV} Let $f\colon [a,b]\to\R$ and $[c,d]$ be a closed subinterval of $[a,b]$. Let $P=\{x\_i : 0\le i\le n\} $ be a partition of $[c,d]$. The $P$-\df{variation} of $f$ over $[c,d]$, denoted by $P\_v(f,[c,d])$, is defined to be the real number \[P\_v(f,[c,d])=\sum\limits\_1^n \abs{f(x\_{i})-f(x\_{i-1})}.\]

\end{definition}

\begin{definition}\label{V} Let $f\colon [a,b]\to\R$ and let $[c,d]$ be a closed subinterval of $[a,b]$. The \df{variation} of $f$ over $[c,d]$, denoted by $V(f,[c,d])$, is defined by

\[V(f,[c,d])=\sup\{P\_v(f,[c,d]): P\text{ is a partition of }[c,d]\},\]

and the function $f$ is said to be of \df{bounded variation} on $[c,d]$ if $V(f,[c,d])$ is a real number.

\end{definition}

\begin{definition}\label{AC} Let $f\colon I\to\R$, where $I$ is an interval. Then $f$ is \df{absolutely continuous} on $I$ if for every $\eps>0$, there is a $\delta>0$ such that for all sets $S=\{[c\_i,d\_{i}] : 1\le i\le n\}$ of non-overlapping sub-intervals of $I$,

\[\text{if } \sum\limits\_1^n \abs{d\_i-c\_i}<\delta, \text{ then } \sum\limits\_1^n \abs{f(d\_{i})-f(c\_i)}<\eps.\]

\end{definition}

\bh

\item Let $f\colon\R\to\R$ be differentiable and let $a\in\R$. Prove that there exists a sequence $\seq{c\_n}$ such that $c\_n\ne a$ for all $n\in\N$ and $\lim\limits\_{n\to\infty}f'(c\_n)=f'(a)$.

%\item Let $A\subseteq\R$ be nonempty. Let $B\subseteq\R$ be the set of all $x\in\R$ such that each neighborhood of $x$ contains at least one element in $A$ and at least one element in the complement of $A$. Prove that $B$ is a closed set.

%\item Let $f\colon [0,1] \to \R$ is a continuous function. Suppose that $f$ is one-to-one and that $f(0)<f(1)$.

%\bhs

%\item Prove that $f(0)<f(x)<f(1)$ for all $x\in(0,1)$.

%\item Prove that $f$ is strictly increasing.

%\ehs

%\item Let $f\colon [0,\infty)\to\R$ be continuous. Suppose that $\lim\limits\_{x\to\infty}f(x)=L$. Evaluate

%\[\lim\limits\_{n\to\infty}\int\_0^2f(n-x)\,dx\]

%and then prove that your answer is correct.

%\item Let $f\colon\R\to\R$ be differentiable at $c$. Prove that $f$ is continuous at $c$.

%\item Let $f\colon\R\to\R$ be a continuous function. Suppose that for every open interval $(a,b)$, we have that $f[(a,b)]$ is also an open interval. Prove that $f$ is one-to-one.

%\item Let $\seq{a\_n}$ be a sequence where and $a\_n>0$ for all $n\ge 1$. Suppose that $\lim\limits\_{n\to\infty}\frac{a\_{n+1}}{a\_n}<1$. Prove that $\lim\limits\_{n\to\infty}a\_n=0$.

%\item Let $D$ be the open interval $(a,b)$ where $a<b$. Suppose that $f\colon D\to\R$ is uniformly continuous. Suppose that $\seq{x\_n}$ is Cauchy sequence where $x\_n\in D$ for all $n\ge 1$. Prove that $\seq{f(x\_n)}$ is a Cauchy sequence.

\item ({\bf Henderson}) Let $f\colon \R\to\R$ be bounded, continuous and strictly increasing. Prove that $f$ is uniformly continuous.

%\item Let $f\colon[0,\infty)\to\R$ be continuous and differentiable on $(0,\infty)$. Suppose $f(0)=0$ and that $f'$ is increasing on $(0,\infty)$. Define a function $g\colon(0,\infty)\to\R$ by $g(x)=\frac{f(x)}{x}$. Prove that $g$ is increasing on $(0,\infty)$.

%\item Let $c>0$ and $a\in\R$ be such that $0<a<1$. Let $\seq{x\_n}$ be a sequence satisfing $\abs{x\_{n+1}-x\_n}\le ca^n$ for all $n\ge 1$. Prove that $\seq{x\_n}$ is a Cauchy sequence.

%\item Let $a$ be a real number such that $0<a<1$. Suppose that $f\colon\R\to\R$ satisfies

%\[\abs{f(x)-f(y)}\le a\abs{x-y}.\]

%\bhs

%\item Prove that $f$ is continuous.

%\item Define the sequence $\seq{x\_n}$ by $x\_1=f(0)$ and $x\_{n+1}=f(x\_n)$ for all $n\ge 1$. Prove that $\abs{x\_{n+1}-x\_n}\le\abs{f(0)}a^n$ for all $n\ge 1$.

%\item Prove that there is an $\ell\in\R$ such that $f(\ell)=\ell$.

%\ehs

%\item Suppose $f\colon\R\to\R$ is differentiable. Prove that if $\lim\limits\_{x\to\infty}f'(x)=0$, then $\lim\limits\_{x\to\infty}[f(x+1)-f(x)]=0$.

%\item Let $f\colon (a,b)\to \R$ be a differentiable function on the open interval $(a,b)$. Suppose

%for some $M\ge 0$ we have $\abs{f'(x)}\le M$ for all $x\in (a,b)$. Prove that $f$ is uniformly continuous on $(a,b)$.

\item ({\bf Miller}) Let $f\colon [a,b]\to \R$ be a continuous function on the closed interval $[a,b]$ where $f(a)<0<f(b)$. Let $S=\{x\in [a,b] : f(x)<0\}$, and let $u=\sup(S)$. Prove that $f(u)=0$.

%\item Suppose that $\sum\_{n=1}^\infty a\_n$ converges absolutely. Prove that $\sum\_{n=1}^\infty a\_n^2$ converges.

\item Suppose $f\colon \R \to \R$ is a differentiable function and there is no $x\in \R$ such that $f(x) = 0 = f'(x)$.

Let $Z\_f= \{x\in \R : f(x) = 0\}$ be the zero set of $f$. Prove that $Z\_f$ has no accumulation points.

%\item

%\bhs

%\item Suppose that $a<c<d<b$. Let $f\colon [a,b]\to \R$ be defined by

%\[f(x)=\begin{cases}

%1, &\text{if $c<x<d$;}\\

%0, &\text{if $x\notin (c,d)$.}

%\end{cases}\]

%Use Theorem \ref{thm29.9}, of the real analysis notes, to prove that $f$ is Riemann integrable over the interval $[a, b]$.

%\item Let $f\colon[a,b]\to \R$ be continuous and suppose for every integrable function $g\colon[a,b]\to \R$, we have that $\int\_a^bfg=0$. Prove that for all $x\in[a,b]$ we have that $f(x)=0$.

%\ehs

%\item State the alternating series test for convergence of an infinite series. Prove that a series satisfying the conditions of this test is convergent.

\item Suppose $\lim\limits\_{n\to\infty}s\_n=c$ and let $\sigma\colon \N\to\N$ be one-to-one where $\N=\{1,2,3,4,5,\dots\}$ is the set of natural numbers. Prove that $\lim\limits\_{n\to\infty}s\_{\sigma(n)}=c$.

%\item Suppose that $f\colon D\to\R$ preserves convergent sequences. Prove that $f$ is continuous.

\item Let $a<x\_0<b$ and suppose that $f\colon (a,b)\to\R$ is differentiable. Prove the following:

\bhs

\item For all $\eps>0$ and $\delta>0$ there is a $c\in(a,b)$ so that $0<\abs{c-x\_0}<\delta$ and $\abs{f'(c)-f'(x\_0)}<\eps$.

\item If $\lim\limits\_{x\to x\_0}f'(x)=L$, then $f'(x\_0)=L$. [Hint: Prove that $\abs{f'(x\_0)-L}<\eps$ for all $\eps>0$.]

\ehs

\item Let $a< b$. Suppose that $F\colon [a,b]\to\R$ and $f\colon [a,b]\to\R$ are continuous. If $F'(x)=f(x)$ for all $x$ in $(a,b]$, then $F'(a)=f(a)$. (The derivative at an endpoint is the appropriate one-sided limit of the difference quotient.)

\item Let $f\colon(0,1]\to\R$ be differentiable on $(0,1]$. Suppose that $\abs{f'(x)}\le 1$ for all $x\in (0,1]$. Define a sequence $\seq{t\_n}$ by $t\_n=f(\frac{1}{n})$ for all $n\ge 1$. Prove that $\seq{t\_n}$ converges.

\item Suppose that $f\colon D\to\R$ preserves convergent sequences. Prove that $f$ is continuous.

\item Suppose that $f\colon D\to\R$ preserves convergent sequences. Prove that if $D$ is bounded, then $f$ is uniformly continuous.

%\item Suppose that $f\colon\R\to\R$ is continuous. Prove that

%\[\int\_0^xf(u)(x-u)\, du =\int^x\_0\left(\int^u\_0f(t)\, dt\right)du.\]

\item Let $f\colon[a,b]\to\R$ be a continuous function that is differentiable on $(a,b)$. Suppose that $f' \colon(a,b)\to\R$ is bounded. Prove that $f$ is of bounded variation on $[a,b]$.

\item Let $f\colon[a,b]\to\R$ be a continuous function that is differentiable on $(a,b)$. Suppose that $f' \colon(a,b)\to\R$ is bounded. Prove that $f$ is absolutely continuous on $[a,b]$.

\item Let $I$ be an interval. Suppose that $f\colon I\to\R$ is absolutely continuous. Prove that $f\colon I\to\R$ is uniformly continuous.

\item Let $I$ be an interval. Suppose that $f\colon I\to\R$ is absolutely continuous. Prove that $\abs{f}$ is absolutely continuous.

\item Let $I$ be an interval. Suppose that $f\colon I\to\R$ and $g\colon I\to\R$ are absolutely continuous. Prove that $f+g$ is absolutely continuous.

\item ({\bf Wood}) Let $[a,b]$ be an interval. Suppose that $f\colon [a,b]\to\R$ and $g\colon [a,b]\to\R$ are of bounded variation. Prove that $f+g$ is of bounded variation.

\item Suppose that $f\colon \R\to\R$ is differentiable and $\abs{f'(x)}<1$ for all $x\in\R$. Prove that $f$ has at most one fixed point. [Recall that $c$ is a fixed point of $f$ when $f(c)=c$.]

\item Let $a<b$, $I=[a,b)$, and $f\colon I\to\R$ be differentiable on $I$ with $\abs{f'(x)}\le1$ for all $x\in I$. Suppose that $\seq{x\_i}$ is a sequence of distinct points in $I$ that converges to $b$. Prove that sequence $\seq{f(x\_i)}$ converges.

\eh

\section{Group Theory Problems}

\bh

%\item Let $G$ be a group. Define $Z(G)$, the center of $G$, to be $Z(G)=\{x\in G: xy =yx \text{ for all $y\in G$}\}.$ Prove that $Z(G)$ is a subgroup of $G$.

%\item Let $G$ be an abelian group with an element of order greater than $2$. Show that there is an automorphism of $G$ that is different than the identity automorphism.

%\item Let $G$ be a non-trivial abelian finite group (that is, there is an $a\in G$ such that $a\ne e$, where $e$ is the identity element of $G$). Suppose that each $a\in G$ has even order whenever $a\ne e$. Let $n$ be any integer. Define the function $\varphi\colon G\to G$ by $\varphi(x)=x^n$ for each $x\in G$. Prove that $\varphi$ is an automorphism if and only if $n$ is odd.

%\item Let $G$ be a group and let $H\subseteq G$. Define the relation $\sim$ on $G$ by \[x\sim y \text{ if and only if } xy^{-1}\in H.\]

%\bhs

%\item Suppose that $H$ is a subgroup of $G$. Prove that $\sim$ is an equivalence relation on $G$.

%\item Suppose that $\sim$ is an equivalence relation on $G$. Prove that $H$ is a subgroup of $G$.

%\ehs

\item ({\bf Downing}) Let $G$ be a finite group and let $g$ and $h$ be a elements in $G$. Suppose the positive natural number $m$ is the smallest such that $h^m$ commutes with $g$. Prove that $m$ evenly divides the order of $h$.

%\item Let $G$ be an abelian group. Let $H\subseteq G$

% be defined by $H=\{a^3 : a\in G\}$. Prove that $H$ is a subgroup of $G$.

%\item Let $G$ be a group with subgroups $A$ and $B$. Define \[AB=\{x

%: x=ab \text{ for some $a\in A$ and some $b\in B$}\}\]

%and

%\[BA=\{x

%: x=ba \text{ for some $b\in B$ and some $a\in A$}\}.\]

%Suppose that

%$BA\subseteq AB$. Prove that $AB$ is a subgroup.

%\item Let $G$ and $G'$ be groups and let $\varphi\colon G\to G'$ be a homomorphism onto $G'$. Prove that $G'$ is a abelian if and only if $aba^{-1}b^{-1}\in \ker(\varphi)$ for all $a\in G$ and all $b\in G$.

\item Let $G$ be a group and let $Z(G)$ be the center of $G$. Let $a, b\in G$ be a

distinct elements. Define the automorphism $\varphi:G\to G$ by

\[\varphi(x)=a^{-1}xa, \text{ for all $x\in G$}.\]

Now, define the automorphism $\sigma:G\to G$ by

\[\sigma(x)=b^{-1}xb, \text{ for all $x\in G$}.\]

Prove that $\varphi=\sigma$ if and only if $ba^{-1}\in Z(G)$.

\item ({\bf Cretacci}) Let $\varphi:G\to G'$ be

a homomorphism from the group $G$ to the group

$G'$. Suppose that $N'$ is a {\sl normal\/} subgroup of $G'$.

Define $\varphi^{-1}[N']\subseteq G$ by

\[\varphi^{-1}[N']=\{x\in G : \varphi(x)\in N'\}.\]

\bhs

\item Prove that $\varphi^{-1}[N']$ is a subgroup of $G$.

\item Prove that $\varphi^{-1}[N']$ is a {\sl normal\/} subgroup of

$G$.

\ehs

\item ({\bf Williams}) Let $\varphi\colon G\to G'$ be a homomorphism where $G$ and $G'$ are groups. Let $K=\ker(\varphi)$. Let $H$ be a subgroup of $G$. Prove that $\varphi^{-1}[\varphi[H]] =HK$ where $HK=\{hk : h\in H \text{ and } k\in K\}$.

\item Let $n>0$ be a fixed natural number, $G$ be a group and let $e$ be the identity element in $G$. Define $H=\{g\in G : g^n =e\}$. Prove that if $H$ is a subgroup of $G$, then $H$ is normal in $G$.

\item ({\bf Nicholas}) Let $G$ be a group and let $N$ be a normal subgroup of $G$. Suppose that the quotient group $G/N$ has order $m$. Prove that $a^m\in N$ for all $a\in G$.

%\item Let $G$ be the group of one-to-one and onto functions $\tau\colon \Z\to\Z$, with composition as the group operation. Let $H\subseteq G$ be defined by

%\[H=\{\sigma\in G : \sigma(n)=n \text{ for all $n\le 0$}\}.\]

%Prove that $H$ is a subgroup of $G$.

%Let $\alpha\colon \Z\to\Z$ be the one-to-one and onto function defined by $\alpha(n)=n+1$. Show that $\alpha H\alpha^{-1}\subseteq H$ and $\alpha H\alpha^{-1}\ne H$.

%\item Let $(\Q,+)$ be the additive group of rational numbers. Let $(\Z,+)$ be the additive group of integers. Note the $\Z$ is a subgroup of $\Q$. Show that the quotient group $\Q/\Z$ is infinite, but each element in $\Q/\Z$ has finite order.

%\newpage

%\item Let $G$ be an abelian group and let $H=\{x\in G : x \text{ has finite order}\}$.

%\bhs

%\item Prove that $H$ is a subgroup of $G$.

%\item Prove that the only element in the quotient group $G/H$ of finite order is the identity element.

%\ehs

%\item Let $H$ and $K$ be normal subgroups of a group $G$. Suppose that $H\cap K=\{e\}$. Prove that $hk=kh$ for all $h\in H$ and all $k\in K$.

%\item Let $G$ be a group and let $a\in G$ be a

%fixed element. Let $\varphi:G\to G$ be the automorphism defined by

%\[\varphi(x)=a^{-1}xa, \text{ for all $x\in G$.}\]

%Let $N$ be a normal subgroup of $G$. For each $n\in N$ prove that $\varphi(n)\in N$.

%\item Recall that $\N=\{1,2,3,\dots\}$. Let $G$ be a group and let $H$ be a subgroup of $G$. Let $n\in\N$ and $x\in G\setminus H$ be such

% that $x^n\in H$ and for all $k\in\N$ if $k<n$ then $x^k\notin H$. Suppose that $m\in\N$ satisfies $x^m\in H$. Prove that $n\mid m$.

\item Assume that the group $G$ has a subgroup of order $n$, a fixed natural number. Let $\{H\_i : i\in I\}$ be an indexed set consisting of all the subgroups of $G$ of order $n$. Given that $\bigcap\limits\_{i\in I}H\_i$ is a subgroup of $G$. Prove that $\bigcap\limits\_{i\in I}H\_i$ is a normal subgroup of $G$.

%\item Let $G$ be a group and let $H$ and $K$ be subgroups of $G$. Prove that if $G=H\cup K$, then either $G=H$ or $G=K$.

%\item Let $G$ be a group and let $N$

%be a {\sl normal\/} subgroup of $G$. Prove that for any given $a\in G$, the

%following hold:

%\bhs

%\item For all $n\in N$ there exists an $j\in N$ such that $na=aj$.

%\item For all $n\in N$ there exists an $k\in N$ such that $an=ka$.

%\ehs

\item ({\bf Krupa}) Let $G$ be a group and let $N$ and $K$

be {\sl normal\/} subgroups of $G$ such that $N\cap K=\{e\}$, where $e$ is the identity element in $G$. Let $h\in H$ and $k\in K$. Prove that $hk=kh$.

\item Let $\{N\_i : i\in I\}$ be an indexed set consisting of normal subgroups of a group $G$.

\bhs

\item Prove that $\bigcap\limits\_{i\in I}N\_i$ is a subgroup of $G$.

\item Prove that $\bigcap\limits\_{i\in I}N\_i$ is a normal subgroup of $G$.

\ehs

\item ({\bf Edan}) Let $G$ be a group and let $\varphi\colon G\to G'$ be a homomorphism where $G'$ is an abelian group. Prove that if $K$ is a subgroup of $G$ such that $\ker(\varphi)\subseteq K$, then $K$ is normal in $G$.

%\item Prove that any non-abelian group has an automorphism that is different from the identity automorphism.

%\item Let $G$ be a group and let $H$ be a finite subgroup of $G$ of order $n$. Suppose that $H$ is the only subgroup of $G$ of order $n$. Prove that $H$ is a normal subgroup of $G$.

%\item Let $G$ be a finite group and for any $x\in G$ let $\o(x)$ denote the order of $x$. Prove the following:

%\bhs

%\item Let $a\in G$ and $b\in G$. Then $\o(a)=\o(b)$ if and only if for all $n\in\N$, $a^n=e$ iff $b^n=e$.

%\item $\o(a)=\o(a^{-1})$, for any $a\in G$.

%\item $\o(ab)=\o(ba)$, for any $a$ and $b$ in $G$.

%\ehs

%\item Let $G$ be a group and let $f\colon G\to G$ be defined by $f(a)=a^{-1}$. Prove that $G$ is abelian if and only if $f$ is a homomorphism.

%\item Let $G$ be a group and let $a\in G$ be a

%fixed element. Define the function $\varphi:G\to G$ by

%\[\varphi(x)=a^{-1}xa, \ \text{ for all $x\in G$.}\]

%\bhs

%\item Prove that $\varphi\colon G\to G$ is a homomorphism.

%\item Prove that $\varphi\colon G\to G$ is one-to-one.

%\item Prove that $\varphi\colon G\to G$ is onto.

%\ehs

\eh

%\subsection\*{Ring Theory Problems}

%\bh

%\item Let $R$ be a finite commutative ring with a unity element. Show that an element $r\in R$ is invertible if and only if $r^n=1$ for some $n>0$.

%\item Let $\Z[i]$ be the ring of Gaussian integers and let $\Z\_5$ be the ring of congruence classes $(\text{mod } 5)$. Define $\varphi\colon \Z[i]\to \Z\_5$ by $\varphi(a+ib)=[a-2b]\_5$. Prove that $\varphi$ is a ring homomorphism.

%\item Let $R$ be a commutative ring. Let $I$ be an ideal in $R$. Define $J\subseteq R$ by \[J=\{r\in R : r^n\in I \text{ for some positive integer $n$}\}.\] Prove that $J$ is an ideal in $R$. [Hint: The binomial formula applies to any commutative ring. Thus, for each $a,b\in R$, we have that $(a+b)^{n+m-1}=\sum\_{i=0}^{n+m-1}\binom{n+m-1}{i}a^ib^{n+m-1-i}$. For each such $i$ show that exactly one of the following is true: (i) $i\ge n$ or (ii) $n + m - 1 - i \ge m$.]

%\item Let $R$ be a ring. An element $x\in R$ is said to be {\it nilpotent\/} if $x^n=0$ for some natural number $n\ge 1$. Let $I=\{x\in R : x \text{ is nilpotent}\}$.

%\bhs

%\item Prove that $x+y\in I$ for all $x,y\in I$.

%\item Prove that $I$ is an ideal in $R$.

%\item Prove that only element in the quotient ring $R/I$ that is nilpotent is the zero element.

%\ehs

%\item The minimal polynomial $p(x)\in\Q[x]$ of a real number $\alpha$ is the polynomial of least degree with leading coefficient 1 such that $p(\alpha)=0$. Prove that $x^3-2$ is the minimal polynomial in $\Q[x]$ of $\sqrt[3]{2}$.

%\item Let $\Z[x]$ be the ring of polynomials over the integers $\Z$. Prove that the ideal $\seq{x}$ is a prime ideal but is not a maximal ideal.

%\item Let $\N=\{1,2,3,\dots\}$ be the set of natural numbers. Let $\F\_i=(F\_i, +,\cdot)$ be a field for each $i\in\N$. Suppose each $\F\_i$ is a subfield of $\F\_{i+1}$ for each $i\in\N$. Let $\displaystyle F=\bigcup\_{i\in\N}F\_i$. Define an addition operation $+$ on $F$ and a multiplication operation $\cdot$ on $F$. Prove that $(F, +,\cdot)$ is a field.

%\eh

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