# Final Problems in Real Analysis & Abstract Algebra
# MAT 491–Spring 2019

## 1 Real Analysis Problems

**Definition 1.** We shall say that $f\colon D \to \mathbb{R}$ **preserves convergent sequences** if for all convergent sequences $\langle x_n \rangle$ with $x_n \in D$ for all $n \geq 1$, we have that $\langle f(x_n) \rangle$ also converges.

**Definition 2** (Partition of an Interval)**.** Let $[c, d]$ be an interval.

- A finite set $P = \{x_i : 0 \leq i \leq n\} = \{x_0, x_1, x_2, \ldots, x_n\}$ of points in $[a, b]$ is called a **partition** of $[a, b]$ provided that $c = x_0 < x_1 < x_2 < \cdots < x_n = d$.
- If $P$ and $P^*$ are two partitions of $[c, d]$ with $P \subseteq P^*$, then $P^*$ is called a **refinement** of $P$.

**Remark.** Given two partitions $P$ and $Q$ of $[c, d]$, it follows that $P \cup Q$ is a partition of $[c, d]$ and so, $P \cup Q$ is a refinement of both $P$ and $Q$.

**Definition 3.** Let $f\colon [a, b] \to \mathbb{R}$ and $[c, d]$ be a closed subinterval of $[a, b]$. Let $P = \{x_i : 0 \leq i \leq n\}$ be a partition of $[c, d]$. The $P$-**variation** of $f$ over $[c, d]$, denoted by $P_v(f, [c, d])$, is defined to be the real number

$$P_v(f, [c, d]) = \sum_{1}^{n} |f(x_i) - f(x_{i-1})| .$$

**Definition 4.** Let $f\colon [a, b] \to \mathbb{R}$ and let $[c, d]$ be a closed subinterval of $[a, b]$. The **variation** of $f$ over $[c, d]$, denoted by $V(f, [c, d])$, is defined by

$$V(f, [c, d]) = \sup\{P_v(f, [c, d]) : P \text{ is a partition of } [c, d]\},$$

and the function $f$ is said to be of **bounded variation** on $[c, d]$ if $V(f, [c, d])$ is a real number.

**Definition 5.** Let $f\colon I \to \mathbb{R}$, where $I$ is an interval. Then $f$ is **absolutely continuous** on $I$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that for all sets $S = \{[c_i, d_i] : 1 \leq i \leq n\}$ of non-overlapping sub-intervals of $I$,

$$\text{if } \sum_{1}^{n} |d_i - c_i| < \delta, \text{ then } \sum_{1}^{n} |f(d_i) - f(c_i)| < \varepsilon.$$

**1.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable and let $a \in \mathbb{R}$. Prove that there exists a sequence $\langle c_n \rangle$ such that $c_n \neq a$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} f'(c_n) = f'(a)$.

**2.** (**Henderson**) Let $f\colon \mathbb{R} \to \mathbb{R}$ be bounded, continuous and strictly increasing. Prove that $f$ is uniformly continuous.

**3.** (**Miller**) Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$ where $f(a) < 0 < f(b)$. Let $S = \{x \in [a, b] : f(x) < 0\}$, and let $u = \sup(S)$. Prove that $f(u) = 0$.

**4.** Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function and there is no $x \in \mathbb{R}$ such that $f(x) = 0 = f'(x)$. Let $Z_f = \{x \in \mathbb{R} : f(x) = 0\}$ be the zero set of $f$. Prove that $Z_f$ has no accumulation points.

**5.** Suppose $\lim_{n\to\infty} s_n = c$ and let $\sigma\colon \mathbb{N} \to \mathbb{N}$ be one-to-one where $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$ is the set of natural numbers. Prove that $\lim_{n\to\infty} s_{\sigma(n)} = c$.

**6.** Let $a < x_0 < b$ and suppose that $f\colon (a, b) \to \mathbb{R}$ is differentiable. Prove the following:

(a) For all $\varepsilon > 0$ and $\delta > 0$ there is a $c \in (a, b)$ so that $0 < |c - x_0| < \delta$ and $|f'(c) - f'(x_0)| < \varepsilon$.

(b) If $\lim_{x\to x_0} f'(x) = L$, then $f'(x_0) = L$. [Hint: Prove that $|f'(x_0) - L| < \varepsilon$ for all $\varepsilon > 0$.]

**7.** Let $a < b$. Suppose that $F\colon [a, b] \to \mathbb{R}$ and $f\colon [a, b] \to \mathbb{R}$ are continuous. If $F'(x) = f(x)$ for all $x$ in $(a, b]$, then $F'(a) = f(a)$. (The derivative at an endpoint is the appropriate one-sided limit of the difference quotient.)

**8.** Let $f\colon (0,1] \to \mathbb{R}$ be differentiable on $(0,1]$. Suppose that $|f'(x)| \leq 1$ for all $x \in (0,1]$. Define a sequence $\langle t_n \rangle$ by $t_n = f(\frac{1}{n})$ for all $n \geq 1$. Prove that $\langle t_n \rangle$ converges.

**9.** Suppose that $f\colon D \to \mathbb{R}$ preserves convergent sequences. Prove that $f$ is continuous.

**10.** Suppose that $f\colon D \to \mathbb{R}$ preserves convergent sequences. Prove that if $D$ is bounded, then $f$ is uniformly continuous.

**11.** Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function that is differentiable on $(a,b)$. Suppose that $f'\colon (a,b) \to \mathbb{R}$ is bounded. Prove that $f$ is of bounded variation on $[a,b]$.

**12.** Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function that is differentiable on $(a,b)$. Suppose that $f'\colon (a,b) \to \mathbb{R}$ is bounded. Prove that $f$ is absolutely continuous on $[a,b]$.

**13.** Let $I$ be an interval. Suppose that $f\colon I \to \mathbb{R}$ is absolutely continuous. Prove that $f\colon I \to \mathbb{R}$ is uniformly continuous.

**14.** Let $I$ be an interval. Suppose that $f\colon I \to \mathbb{R}$ is absolutely continuous. Prove that $|f|$ is absolutely continuous.

**15.** Let $I$ be an interval. Suppose that $f\colon I \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ are absolutely continuous. Prove that $f + g$ is absolutely continuous.

**16.** (**Wood**) Let $[a,b]$ be an interval. Suppose that $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ are of bounded variation. Prove that $f + g$ is of bounded variation.

**17.** Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable and $|f'(x)| < 1$ for all $x \in \mathbb{R}$. Prove that $f$ has at most one fixed point. [Recall that $c$ is a fixed point of $f$ when $f(c) = c$.]

**18.** Let $a < b$, $I = [a,b)$, and $f\colon I \to \mathbb{R}$ be differentiable on $I$ with $|f'(x)| \leq 1$ for all $x \in I$. Suppose that $\langle x_i \rangle$ is a sequence of distinct points in $I$ that converges to $b$. Prove that sequence $\langle f(x_i) \rangle$ converges.

## 2 Group Theory Problems

**1.** (**Downing**) Let $G$ be a group and let $Z(G)$ be the center of $G$. Let $a, b \in G$ be a distinct elements. Define the automorphism $\varphi : G \to G$ by

$$\varphi(x) = a^{-1}xa, \text{ for all } x \in G.$$

Now, define the automorphism $\sigma : G \to G$ by

$$\sigma(x) = b^{-1}xb, \text{ for all } x \in G.$$

Prove that $\varphi = \sigma$ if and only if $ba^{-1} \in Z(G)$.

**2.** (**Cretacci**) Let $\varphi : G \to G'$ be a homomorphism from the group $G$ to the group $G'$. Suppose that $N'$ is a *normal* subgroup of $G'$. Define $\varphi^{-1}[N'] \subseteq G$ by

$$\varphi^{-1}[N'] = \{x \in G : \varphi(x) \in N'\}.$$

(a) Prove that $\varphi^{-1}[N']$ is a subgroup of $G$.
(b) Prove that $\varphi^{-1}[N']$ is a *normal* subgroup of $G$.

**3.** (**Williams**) Let $\varphi\colon G \to G'$ be a homomorphism where $G$ and $G'$ are groups. Let $K = \ker(\varphi)$. Let $H$ be a subgroup of $G$. Prove that $\varphi^{-1}[\varphi[H]] = HK$ where $HK = \{hk : h \in H \text{ and } k \in K\}$.

**4.** Let $n > 0$ be a fixed natural number, $G$ be a group and let $e$ be the identity element in $G$. Define $H = \{g \in G : g^n = e\}$. Prove that if $H$ is a subgroup of $G$, then $H$ is normal in $G$.

**5.** (**Nicholas**) Let $G$ be a group and let $N$ be a normal subgroup of $G$. Suppose that the quotient group $G/N$ has order $m$. Prove that $a^m \in N$ for all $a \in G$.

**6.** Assume that the group $G$ has a subgroup of order $n$, a fixed natural number. Let $\{H_i : i \in I\}$ be an indexed set consisting of all the subgroups of $G$ of order $n$. Given that $\bigcap_{i \in I} H_i$ is a subgroup of $G$. Prove that $\bigcap_{i \in I} H_i$ is a normal subgroup of $G$.

**7.** (**Krupa**) Let $G$ be a group and let $N$ and $K$ be *normal* subgroups of $G$ such that $N \cap K = \{e\}$, where $e$ is the identity element in $G$. Let $h \in H$ and $k \in K$. Prove that $hk = kh$.

**8.** Let $\{N_i : i \in I\}$ be an indexed set consisting of normal subgroups of a group $G$.

(a) Prove that $\bigcap_{i \in I} N_i$ is a subgroup of $G$.

(b) Prove that $\bigcap_{i \in I} N_i$ is a normal subgroup of $G$.

**9.** (**Edan**) Let $G$ be a group and let $\varphi\colon G \to G'$ be a homomorphism where $G'$ is an abelian group. Prove that if $K$ is a subgroup of $G$ such that $\ker(\varphi) \subseteq K$, then $K$ is normal in $G$.