MAT 491, Spring 2019
Problems in Real Analysis & Abstract Algebra
Select Only One Problem – First Come, First Serve

1 Real Analysis Problems – Only for students who have passed MAT 417

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable and assume that $f'\colon \mathbb{R} \to \mathbb{R}$ is continuous. Let $a < b < c$ be such that $f(a) < f(b)$ and $f(c) < f(b)$. Prove that there is an $x \in [a, c]$ such that $f'(x) = 0$.

2. (**Miller**) Let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be functions. Suppose g is continuous at $c \in D$ and $|f(x) - f(c)| \leq |g(x)|\,|x - c|$ for all $x \in D$. Prove that f is continuous at c.

3. Let D be the open interval (a, b) where $a < b$. Suppose that $f\colon D \to \mathbb{R}$ is uniformly continuous. Suppose that $\langle x_n \rangle$ is Cauchy sequence where $x_n \in D$ for all $n \geq 1$. Prove that $\langle f(x_n) \rangle$ is a Cauchy sequence.

4. Let $c > 0$ and $a \in \mathbb{R}$ be such that $0 < a < 1$. Let $\langle x_n \rangle$ be a sequence satisfying $|x_{n+1} - x_n| \leq ca^n$ for all $n \geq 1$. Prove that $\langle x_n \rangle$ is a Cauchy sequence.

5. Suppose that $D \subseteq \mathbb{R}$ and that $f\colon D \to \mathbb{R}$ is uniformly continuous. Let $\langle x_n \rangle$ be a sequence of points in D. Prove that if $\langle x_n \rangle$ converges, then $\langle f(x_n) \rangle$ is a Cauchy sequence.

6. Suppose that $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ are uniformly continuous. Prove that $f + kg$ is uniformly continuous.

7. Let $f\colon D \to \mathbb{R}$ and $g\colon E \to \mathbb{R}$ be functions such that $f[D] \subseteq E$. Suppose that f and g are uniformly continuous. Prove that $(g \circ f)\colon D \to \mathbb{R}$ is uniformly continuous.

8. Suppose that $f\colon (0, 1) \to \mathbb{R}$ is uniformly continuous. Prove that $\lim\limits_{n\to\infty} f(\frac{1}{n})$ exists.

9. (**Edan**) Let I be an interval and f be a differentiable function on I. Suppose that $f(x) \leq f(y)$ whenever $x \leq y$ are in I. Prove that $f'(c) \geq 0$ for all $c \in I$. Let $f\colon [0, 2] \to \mathbb{R}$ be continuous on $[0, 2]$ and differentiable on $(0, 2)$. Suppose that $f(0) = 0$ and $f(1) = f(2) = 1$.

(a) Show that there is a $c \in (0, 1)$ such that $f'(c) = 1$.
(b) Show that there is a $c \in (1, 2)$ such that $f'(c) = 0$.
(c) Show that there is a $c \in (0, 2)$ such that $f'(c) = \frac{1}{3}$.

10. (**Wood**) Let $f\colon I \to \mathbb{R}$ be a function where I is an interval. Suppose that f is differentiable on I. Prove that if f is increasing on I, then $f'(x) \geq 0$ for all $x \in I$.

2 Group Theory Problems – Only for students who have passed MAT 301

1. Let G be an abelian group with an element of order greater than 2. Show that there is a one-to-one homomorphism $h\colon G \to G$ that is different than the identity homomorphism.

2. (**Krupa**) Let G be an abelian group. Let $H \subseteq G$ be defined by $H = \{a^3 : a \in G\}$. Prove that H is a subgroup of G.

3. (**Cretacci**) Let G and G' be groups and let $\varphi\colon G \to G'$ be a homomorphism onto G'. Prove that G' is a abelian if and only if $aba^{-1}b^{-1} \in \ker(\varphi)$ for all $a \in G$ and all $b \in G$.

4. Let G be a group with subgroups A and B. Define

$$AB = \{x : x = ab \text{ for some } a \in A \text{ and some } b \in B\}$$

and

$$BA = \{x : x = ba \text{ for some } b \in B \text{ and some } a \in A\}.$$

Suppose that $BA \subseteq AB$. Prove that AB is a subgroup of G.

5. Let G be a group and let H and K be subgroups of G. Prove that if $G = H \cup K$, then either $G = H$ or $G = K$.

6. (**Downing**) Let G be a group and let $f\colon G \to G$ be defined by $f(a) = a^{-1}$. Prove that G is abelian if and only if f is a homomorphism.

7. Let G be a group and let $a \in G$ be a fixed element. Define the function $\varphi : G \to G$ by

$$\varphi(x) = a^{-1}xa, \quad \text{for all } x \in G.$$

   (a) Prove that $\varphi\colon G \to G$ is a homomorphism.
   (b) Prove that $\varphi\colon G \to G$ is one-to-one.
   (c) Prove that $\varphi\colon G \to G$ is onto.

8. Let G be a finite group and let g and h be elements in G. Suppose the positive natural number m is the smallest such that h^m commutes with g. Prove that m evenly divides the order of h.

9. Let $n > 0$ be a fixed natural number, G be a group and let e be the identity element in G. Define $H = \{g \in G : g^n = e\}$. Prove that if H is a subgroup of G, then H is normal in G.

10. Prove that any non-abelian group has an automorphism that is different from the identity automorphism.

11. (**Nicholas**) Let G be a group and let H be a subgroup of G. Let $a \in G$ and let $J = \{a^{-1}ha : h \in H\}$. Prove that J is a subgroup of G.

12. (**Williams**) Let G be a group and let H to be $H = \{x \in G : xy = yx \text{ for all } y \in G\}$. Prove that H is a subgroup of G.