MAT 491, Spring 2019 – Problem Set in Linear Algebra

STATE YOUR PROBLEM AS A THEOREM, AND COMPOSE A PROOF OF THE THEOREM.

1. Let $T\colon V \to V$ be a linear transformation where V is a vector space over $\mathbb{R}$. Let $k > 0$ be a natural number such that $T^k(\mathbf{v}) = \boldsymbol{\theta}$ for all $\mathbf{v} \in V$, were $\boldsymbol{\theta} \in V$ is the zero vector. Suppose that there is a nonzero vector $\mathbf{x} \in V$ such that $T^{k-1}(\mathbf{x}) \neq \boldsymbol{\theta}$. Prove that $\{\mathbf{x}, T(\mathbf{x}), \ldots, T^{k-1}(\mathbf{x})\}$ is a linearly independent set of vectors. [Recall that T^k denotes repeated composition of T with itself, for example, $T^2(\mathbf{v}) = T(T(\mathbf{v}))$.]

2. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a self-adjoint linear transformation. Let W be a subspace of $\mathbb{R}^n$ that is invariant under T (that is, $T[W] \subseteq W$). Show that $W^\perp$ is invariant under T, where $W^\perp = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{y} = 0 \text{ for all } \mathbf{y} \in W\}$. [$T$ is *self-adjoint* if $T(\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, where $\cdot$ is the dot product.]

3. Let A, B and S be $n \times n$ matrices where S is invertible. Suppose that $S^{-1}AS = B$.

(a) Prove that A and B have the same eigenvalues.

(b) Let λ be an eigenvalue for A and for B. Let E^A_λ be the eigenspace associated with the matrix A and let E^B_λ be the eigenspace associated with the matrix B. Prove that these two eigenspaces have the same dimension.

4. (**Ibrahim**) Let A, B and S be $n \times n$ matrices where S is invertible. Suppose that $S^{-1}AS = B$. Prove that the null space of A and the null space of B have the same dimension. Conclude that $\operatorname{rank}(A) = \operatorname{rank}(B)$. Let $F\colon V \to V$, $G\colon W \to W$, and $H\colon V \to W$ be linear transformations, where V and W are vector spaces. Suppose $H \circ F = G \circ H$ and that H is a bijection. Prove that all of the eigenvalues of G are eigenvalues of F.

5. (**Cretacci**) Let A, B and S be $n \times n$ matrices where S is invertible. Suppose that $S^{-1}AS = B$. Prove that the null space of A and the null space of B have the same dimension. Conclude that $\operatorname{rank}(A) = \operatorname{rank}(B)$.

6. Let $T\colon V \to V$ be a linear transformation where V is a vector space over $\mathbb{R}$. Suppose that $k > 0$ is a natural number such that $T^k(\mathbf{v}) = \boldsymbol{\theta}$ for all $\mathbf{v} \in V$, were $\boldsymbol{\theta} \in V$ is the zero vector. Let $F\colon V \to V$ be the linear transformation defined by $F(\mathbf{v}) = \mathbf{v} + T(\mathbf{v})$. Prove that F is one-to-one. [Recall that T^k denotes repeated composition of T with itself, for example, $T^2(\mathbf{v}) = T(T(\mathbf{v}))$.]

7. Let V and W be vector spaces. Suppose that $T\colon V \to W$ is a linear transformation.

(a) Assume that T is one-to-one. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be linearly independent vectors in V. Prove that $T(\mathbf{x}_1), T(\mathbf{x}_2), \ldots, T(\mathbf{x}_k)$ are linearly independent vectors in W.

(b) Assume that T is onto W. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be vectors in V. Prove if $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ span V, then $T(\mathbf{x}_1), T(\mathbf{x}_2), \ldots, T(\mathbf{x}_k)$ span W.

8. (**Edan**) Let $\mathcal{P}$ denote the vector space of all polynomials with real coefficients and degree $\leq n$, with the usual polynomial addition and scalar multiplication. Define the function $T\colon \mathcal{P} \to \mathcal{P}$ by $T(p(x)) = p(x) + p'(x)$.

(a) Prove that T is a linear transformation.

(b) Prove that T is one-to-one.

(c) Prove that T is onto $\mathcal{P}$.

9. (**Downing**) Let A be an $n \times n$ matrix and let $\mathbf{v}$ be an eigenvector for A with eigenvalue λ.

(a) Let $B = P^{-1}AP$ for an invertible $n \times n$ matrix P. Find an eigenvector and associate eigenvalue for B.

(b) Let $f(x)$ be a polynomial of degree k and let $C = f(A)$. Find an eigenvector and associate eigenvalue for C. [For example, suppose that $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$. Then $f(A) = a_3A^3 + a_2A^2 + a_1A + a_0I$ where I is the $n \times n$ identity matrix.]

10. (**Williams**) Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for $\mathbb{R}^n$. Prove that T is one-to-one if and only if $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\}$ is also a basis for $\mathbb{R}^n$.

11. (**Krupa**) Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{x}_{k+1}, \ldots \mathbf{x}_n\}$ be a basis for $\mathbb{R}^n$ so that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for the null space of T. Prove that $\{T(\mathbf{x}_{k+1}), \ldots T(\mathbf{x}_n)\}$ is a basis for the image space $T[\mathbb{R}^n]$.

12. Let $\mathcal{P}_3$ be the vector space of polynomials with real coefficients and degree ≤ 3 and let $\mathcal{P}_2$ be the vector space of polynomials with real coefficients and degree ≤ 2. Consider the function $T : \mathcal{P}_2 \to \mathcal{P}_3$ defined by $T(p) = \int_0^x p(x)\,dx$. Prove that T is a linear transformation. Then prove that T is one-to-one. Finally, show that T is not onto $\mathcal{P}_3$.

13. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{x}_{k+1}, \ldots \mathbf{x}_n\}$ be a basis for $\mathbb{R}^n$ so that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for the null space of T. Prove that $\{T(\mathbf{x}_{k+1}), \ldots T(\mathbf{x}_n)\}$ is a basis for the image space $T[\mathbb{R}^n]$.

14. Let $T\colon V \to W$ be a linear transformation from vector space V to vector space W. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a finite set of vectors each of which is in V.

(a) Prove that if $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ is linearly independent, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent.

(b) Suppose that T is one-to-one. Prove that if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent, then the set of vectors $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ is linearly independent.

15. Let $T\colon V \to V$ be a linear transformation where V is a vector space over $\mathbb{R}$. Suppose that $T^4(\mathbf{v}) = \boldsymbol{\theta}$ for all $\mathbf{v} \in V$, were $\boldsymbol{\theta} \in V$ is the zero vector. Let $F\colon V \to V$ be the linear transformation defined by $F(\mathbf{v}) = \mathbf{v} + T(\mathbf{v})$. Prove that F is onto V. [Recall that T^k denotes repeated composition of T with itself, for example, $T^2(\mathbf{v}) = T(T(\mathbf{v}))$.]

16. Let V and W be vector spaces, and let $T\colon V \to W$ be a surjective linear transformation. Suppose that $A \subseteq V$ is a set of vectors that spans V (A may be an infinite set). Let $B \subseteq W$ be defined by $B = \{T(\mathbf{v}) : \mathbf{v} \in A\}$. Prove that B spans W.

17. (**Wood**) Let V and W be vector spaces. Let $T\colon V \to W$ be a one-to-one and onto linear transformation. Thus, the inverse function of T, denoted by $T^{-1}\colon W \to V$, exists. Prove that $T^{-1}\colon W \to V$ is a linear transformation.

18. (**Budd-Nicholas**) Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that every nonzero $\mathbf{x} \in \mathbb{R}^n$ is an eigenvector. Show that there exists a real number a such that $T(\mathbf{v}) = a\mathbf{v}$ for all vectors $\mathbf{v} \in \mathbb{R}^n$.

19. (**Miller**) Let A be a nonzero square matrix. Suppose that A^k equals the zero matrix for some natural number $k > 0$. Prove that A is not diagonalizable.

20. (**Henderson**) Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ and $S\colon \mathbb{R}^n \to \mathbb{R}^n$ be linear transformations. Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a basis for $\mathbb{R}^n$. Suppose that each vector in $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is an eigenvector for S and for T. Prove that $ST = TS$, where ST and TS denote compositions.

21. Let $\mathcal{P}_3$ be the vector space of polynomials with real coefficients and degree ≤ 3. Consider the linear transformation $T : \mathcal{P}_3 \to \mathcal{P}_3$ defined by $T(p(x)) = p(x) + p(-x)$. The vectors space $\mathcal{P}_3$ has the basis $x^3, x^2, x, 1$ with the given order (that is, x^3 is the first basis vector, x^2 is the second basis vector, etc.. Thus, every polynomial $p(x) = ax^3 + bx^2 + cx + d$ will be represented by the vector $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$.

(a) Find the 4×4 matrix A that represents T with respect to the basis $x^3, x^2, x, 1$; that is, find the matrix A so that $T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = A\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$.

(b) Find a basis for the null space of T.