```
\documentclass[11pt,oneside,reqno]{amsart}
\usepackage{amsmath}
\usepackage{amsthm}
\usepackage{amssymb}
\usepackage{latexsym}
\usepackage[mathscr]{eucal}
\usepackage{ifthen}
\usepackage{mathtools}
\usepackage{graphicx}
\date{}

%------------------------------- New theorems

% These will be typeset in Roman
\theoremstyle{definition}
\newtheorem{theorem}{Theorem}[section]
\newtheorem{corollary}[theorem]{Corollary}
\newtheorem{proposition}[theorem]{Proposition}
\newtheorem{lemma}[theorem]{Lemma}
\newtheorem{thm}{Theorem}[theorem]
\newtheorem{prop}{Proposition}[theorem]
\newtheorem{lem}{Lemma}[theorem]
\newtheorem{cor}{Corollary}[theorem]
\newtheorem{comment}[theorem]{Comment}
\newtheorem*{prob}{Problem}

% These will be typeset in Roman
\theoremstyle{definition}
\newtheorem{definition}[theorem]{Definition}
\newtheorem*{solution}{\it Solution}
\newtheorem*{rmrk}{Remark}
\theoremstyle{definition}
\newtheorem{example}{Example}
\theoremstyle{definition}
\newtheorem*{clam}{Claim}

%\numberwithin{equation}{section}

%--------------------My Definitions
\newcommand\zero{\boldsymbol{\theta}}
\newcommand\zerom{\mathcal{O}}

\def\x{\mathbf{x}}
\def\y{\mathbf{y}}
\def\u{\mathbf{u}}
\def\v{\mathbf{v}}
\def\P{\mathcal{P}}
\def\F{\mathcal{F}}
\def\G{\mathcal{G}}
\def\K{\mathcal{K}}
\newcommand{\be}{\begin{enumerate}}
\newcommand{\ee}{\end{enumerate}}
\newcommand{\bi}{\begin{itemize}}
\newcommand{\ei}{\end{itemize}}
\newcommand{\bR}{\boldsymbol{\mathbb{R}} }
\newcommand{\R}{\ensuremath{\mathbb{R}} }
\newcommand{\N}{\ensuremath{\mathbb{N}} }
\def\Q{{\text{$\mathbb{Q}$} }}
\def\I{{\text{$\mathbb{I}$} }}
\def\Z{{\text{$\mathbb{Z}$} }}
\def\dom{\hbox{dom}}
\def\ran{\hbox{ran}}
\renewcommand{\iff}{\text{\,\ {\textup{ iff}}\,\ }}
\newcommand{\abs}[1]{\left\lvert#1\right\rvert}
\newcommand{\seq}[1]{\left\langle#1\right\rangle}
\newcommand{\maax}[1]{\textup{max}\!\left\{#1\right\}}
\newcommand{\miin}[1]{\textup{min}\!\left\{#1\right\}}
\newcommand{\set}[1]{\left\{#1\right\}}
\newcommand{\rank}{\textup{rank}}
\newcommand{\df}[1]{{\bf #1}} % highlight the key word in a definition
\newcommand{\txx}[1]{\quad\text{\small{#1}}}
\newcounter{ssbex}
\newcommand{\bhss}{\begin{list}{(\roman{ssbex})}{\usecounter{ssbex}\setlength{\itemsep}{.05\itemsep}\setlength{\topsep}{.5\topsep}}} % beginning of exercise subsublist
\newcommand{\ehss}{\end{list}}
\setlength{\textwidth}{6.5in}
\newlength{\mymargin}
\setlength{\mymargin}{-0.0in} %(8.5in - \textwidth)/2 - 1in
\setlength{\oddsidemargin}{\mymargin}
\setlength{\evensidemargin}{\mymargin}
\setlength{\textheight}{9.5in}
\setlength{\topmargin}{-0.65in} %(10.0in - \textheight)/2 - 1in}
%end-Set Size of paper
\newlength{\myitemsep}
%\setlength{\myitemsep}{\itemsep}
\setlength{\myitemsep}{.5\itemsep}

\newlength{\smitemsep}
\setlength{\smitemsep}{-.3\itemsep}
\newcommand{\bes}{\begin{enumerate}\itemsep=\smitemsep}
\newcommand{\ees}{\end{enumerate}}
\newcommand{\bis}{\begin{itemize}\itemsep=\myitemsep}
\newcommand{\eis}{\end{itemize}}
\newcommand{\biis}{\begin{itemize}\itemsep= \smitemsep}
\newcommand{\eiis}{\end{itemize}}
\newcommand\s{\hspace{.75em}}
%$A=\begin{bmatrix*}[r]
```

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%5&-2\\
%6&-2
%\end{bmatrix*}$
%
%\begin{alignat*}{2}
%A(\x + \y)&= A\x + A\y&&\txx{by distribution of matrix mult.}\\
%&= \zero + \zero&&\txx{by equations in (\ref{quaction}) above}\\
%&= \zero&&\txx{since $\zero + \zero=\zero$.}
%\end{alignat*}
%End of My Definitions
%
\title{A Condition that Ensures Linearly Independent Eigenvectors}
\author{Daniel W. Cunningham}
\address{Department of Mathematics, SUNY Buffalo State}
\email{cunnindw@buffalostate.edu}

\begin{document}
\maketitle
\begin{abstract} A set of linearly independent eigenvectors, of a linear transformation,  can allow one to solve  problems in applied mathematics and in pure mathematics. We present a  condition on a set of eige
guarantees  linearly independence.
\end{abstract}
\section{Introduction}
We first review the relevant definitions and results that we will use in our proof. These topics are typically covered in a first course in linear algebra.
\begin{definition}\label{deflintrans} If  $T\colon V \to W$   is a function from a
vector space  $V$   to the vector space  $W,$  then  $T$   is
called a  linear transformation  if,  for all vectors  $\x$  and $\y$
in $V$    and for all scalars  $c,$   the following hold:
\bes
\item[(a)] $T(\x + \y) = T(\x) + T(\y)$
\item[(b)] $T(c\x) = cT(\x)$.
\ees
The vector space $V$  is called the domain of  $T$   and  the vector space $W$  is called the co-domain of $T.$
\end{definition}
\begin{theorem}\label{ltprop} If  $T\colon V \to W$  is a linear transformation, then
\bes
\item $T(\zero) = \zero$
\item $T(-\x) = -T(\x)$
\item $T(\x-\y) = T(\x)-T(\y)$
\item $T(a\x+b\y) = aT(\x)+bT(\y)$,
\ees
where  $a$ and $b$ are scalars, and $\zero$ denotes the zero vector in $V$ and $W$, respecitively
\end{theorem}
\begin{definition} Let $V$ be a vector space and let $T\colon V \to V$ be a linear transformation. A nonzero vector  $\v$  in $V$   is called an \emph{eigenvector} of $T$   if  $T(\v)$  is a scalar multiple of  $\v$, tl
$T(\v) = \lambda\v$
for some scalar $\lambda$.   The scalar $\lambda$ is called an
{\it eigenvalue\/} of $T$, and the nonzero vector $\x$  is said to be an
{\it eigenvector corresponding} to the eigenvalue $\lambda$.
\end{definition}

\section{Eigenvectors Corresponding to Distinct Eigenvalues are Linearly Independent}
\begin{theorem}Let $T\colon V\to V$ be a linear transformation from a vector space $V$ to itself. Suppose that $T$ has  eigenvectors $\x_1,\x_2,\dots,\x_k$ whose corresponding eigenvalues are all distinct. Th
\x_1,\x_2,\dots,\x_k$ are linearly independent.
\end{theorem}
\begin{proof} Let $T\colon V \to V$ be a linear transformation. Suppose, for a contradiction, that the
 theorem is false for $T$. Thus, there must be a smallest number of eigenvectors $k$, for which the theorem is false. We will now work with this $k$. So $T$ has  eigenvectors $\x_1,\x_2,\dots,\x_k$ and corresp
eigenvalues $\lambda_1, \lambda_2,\dots,\lambda_k$ that are all distinct, that is,
$\lambda_i\ne\lambda_j$ when $i\ne j$ and $1\le i,j\le k$  such that $\x_1,\x_2,\dots,\x_k$ are linearly {\bf de}pendent. First we observe that the eigenvector $\x_1$ is linearly independent. To see this, suppose
Since $\x_1\ne \zero$ (because it is an eigenvector), it follows that $c=0$. Thus, $k>1$. Since $k$ is the smallest for which the theorem is false, it follows that the eigenvectors $\x_1,\x_2,\dots,\x_{k-1}$ are line
dependent.

Since the vectors $\x_1,\x_2,\dots,\x_{k-1},\x_k$ are linearly {\bf de}pendent, there are scalars \[c_1, c_2, \dots, c_{k-1}, c_k,\]which are {\bf not all $0$}, such that
\begin{equation}c_1\x_1+c_2\x_2+ \cdots + c_{k-1}\x_{k-1}+ c_k\x_k=\zero.\label{eigeqat}\end{equation}

We will now show that $c_k\ne 0$. If $c_k= 0$ then we would obtain, from \eqref{eigeqat}, the equation
\begin{equation}c_1\x_1+c_2\x_2+ \cdots + c_{k-1}\x_{k-1}=\zero.\label{eigeqatn}\end{equation}
Because $\x_1,\x_2,\dots,\x_{k-1}$ are linearly {\bf in}dependent, we would be able to conclude from \eqref{eigeqatn} that all of the scalars $c_1, c_2, \dots, c_{k-1}, c_k$ must be $0$, which is not the case. Th
that  $c_k\ne 0$.

Since $c_k\ne 0$ and $\x_k\ne\zero$, equation \eqref{eigeqat} implies that there is a  least one scalar $c_\ell$ such that $c_\ell\ne 0$ where $1\le \ell\le k-1$.
Applying $T$ to both sides of \eqref{eigeqat}, we get
\[c_1T(\x_1)+c_2T(\x_2)+ \cdots + c_{k-1}T(\x_{k-1})+ c_kT(\x_k)=T(\zero).\]
Since $T(\x_i)=\lambda\x_i$, for each $i=1,2,\dots k$, and $T(\zero)=\zero$ by Theorem~\ref{ltprop}(1), we conclude that
\begin{equation}c_1\lambda_1\x_1+c_2\lambda_2\x_2+ \cdots + c_{k-1}\lambda_{k-1}\x_{k-1}+ c_k\lambda_k\x_k=\zero.\label{eigeqat2}\end{equation}
Multiplying both sides of \eqref{eigeqat}  by the scalar $\lambda_k$, we get
\begin{equation}c_1\lambda_k\x_1+c_2\lambda_k\x_2+ \cdots + c_{k-1}\lambda_k\x_{k-1}+ c_k\lambda_k\x_k=\zero.\label{eigeqat3}\end{equation}
Subtracting equation \eqref{eigeqat2} from equation \eqref{eigeqat3}, we see that
\[c_1(\lambda_k-\lambda_1)\x_1+c_2(\lambda_k-\lambda_2)\x_2+ \cdots + c_{k-1}(\lambda_k-\lambda_{k-1})\x_{k-1}=\zero.\]
Because $\x_1,\x_2,\dots,\x_{k-1}$ are linearly independent, we have that
\[c_1(\lambda_k-\lambda_1)=0, \s c_2(\lambda_k-\lambda_2)=0, \s \dots, \s  c_{k-1}(\lambda_k-\lambda_{k-1})=0.\]
Hence, in particular, $c_\ell(\lambda_k-\lambda_\ell)=0$. We noted above that $c_\ell\ne 0$. Thus $\lambda_k-\lambda_\ell=0$ and so, $\lambda_k=\lambda_\ell$. We conclude that not all of the given eigenval
This contradiction completes the proof of the theorem.
\end{proof}\end{document}
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