\documentclass[11pt,oneside,reqno]{amsart}

\usepackage{amsmath}

\usepackage{amsthm}

\usepackage{amssymb}

\usepackage{latexsym}

\usepackage[mathscr]{eucal}

\usepackage{ifthen}

\usepackage{mathtools}

\usepackage{graphicx}

\date{}

%------------------------------- New theorems

% These will be typeset in Roman

\theoremstyle{definition}

\newtheorem{theorem}{Theorem}[section]

\newtheorem{corollary}[theorem]{Corollary}

\newtheorem{proposition}[theorem]{Proposition}

\newtheorem{lemma}[theorem]{Lemma}

\newtheorem{thm}{Theorem}[theorem]

\newtheorem{prop}{Proposition}[theorem]

\newtheorem{lem}{Lemma}[theorem]

\newtheorem{cor}{Corollary}[theorem]

\newtheorem{comment}[theorem]{Comment}

\newtheorem\*{prob}{Problem}

% These will be typeset in Roman

\theoremstyle{definition}

\newtheorem{definition}[theorem]{Definition}

\newtheorem\*{solution}{\it Solution}

\newtheorem\*{rmrk}{Remark}

\theoremstyle{definition}

\newtheorem{example}{Example}

\theoremstyle{definition}

\newtheorem\*{clam}{Claim}

%\numberwithin{equation}{section}

%--------------------My Definitions

\newcommand\zero{\boldsymbol{\theta}}

\newcommand\zerom{\mathcal{O}}

\def\x{\mathbf{x}}

\def\y{\mathbf{y}}

\def\u{\mathbf{u}}

\def\v{\mathbf{v}}

\def\P{\mathcal{P}}

\def\F{\mathcal{F}}

\def\G{\mathcal{G}}

\def\K{\mathcal{K}}

\newcommand{\be}{\begin{enumerate}}

\newcommand{\ee}{\end{enumerate}}

\newcommand{\bi}{\begin{itemize}}

\newcommand{\ei}{\end{itemize}}

\newcommand{\bR}{\boldsymbol{\mathbb{R}} }

\newcommand{\R}{\ensuremath{\mathbb{R}} }

\newcommand{\N}{\ensuremath{\mathbb{N}} }

\def\Q{{\text{$\mathbb{Q}$} }}

\def\I{{\text{$\mathbb{I}$} }}

\def\Z{{\text{$\mathbb{Z}$} }}

\def\dom{\hbox{dom}}

\def\ran{\hbox{ran}}

\renewcommand{\iff}{\text{\,\ {\textup{ iff}}\,\ }}

\newcommand{\abs}[1]{\left\lvert#1\right\rvert}

\newcommand{\seq}[1]{\left\langle#1\right\rangle}

\newcommand{\maax}[1]{\textup{max}\!\left\{#1\right\}}

\newcommand{\miin}[1]{\textup{min}\!\left\{#1\right\}}

\newcommand{\set}[1]{\left\{#1\right\}}

\newcommand{\rank}{\textup{rank}}

\newcommand{\df}[1]{{\bf #1}} % highlight the key word in a definition

\newcommand{\txx}[1]{\quad\text{\small{#1}}}

\newcounter{ssbex}

\newcommand{\bhss}{\begin{list}{(\roman{ssbex})}{\usecounter{ssbex}\setlength{\itemsep}{.05\itemsep}\setlength{\topsep}{.5\topsep}}} % beginning of exercise subsublist

\newcommand{\ehss}{\end{list}}

\setlength{\textwidth}{6.5in}

\newlength{\mymargin}

\setlength{\mymargin}{-0.0in} %(8.5in - \textwidth)/2 - 1in

\setlength{\oddsidemargin}{\mymargin}

\setlength{\evensidemargin}{\mymargin}

\setlength{\textheight}{9.5in}

\setlength{\topmargin}{-0.65in} %(10.0in - \textheight)/2 - 1in}

%end-Set Size of paper

\newlength{\myitemsep}

%\setlength{\myitemsep}{\itemsep}

\setlength{\myitemsep}{.5\itemsep}

\newlength{\smitemsep}

\setlength{\smitemsep}{-.3\itemsep}

\newcommand{\bes}{\begin{enumerate}\itemsep=\smitemsep}

\newcommand{\ees}{\end{enumerate}}

\newcommand{\bis}{\begin{itemize}\itemsep=\myitemsep}

\newcommand{\eis}{\end{itemize}}

\newcommand{\biis}{\begin{itemize}\itemsep= \smitemsep}

\newcommand{\eiis}{\end{itemize}}

\newcommand\s{\hspace{.75em}}

%$A=\begin{bmatrix\*}[r]

%5&-2\\

%6&-2

%\end{bmatrix\*}$

%

%\begin{alignat\*}{2}

%A(\x + \y)&= A\x + A\y&&\txx{by distribution of matrix mult.}\\

%&= \zero + \zero&&\txx{by equations in (\ref{quaction}) above}\\

%&= \zero&&\txx{since $\zero + \zero=\zero$.}

%\end{alignat\*}

%End of My Definitions

%

\title{A Condition that Ensures Linearly Independent Eigenvectors}

\author{Daniel W. Cunningham}

\address{Department of Mathematics, SUNY Buffalo State}

\email{cunnindw@buffalostate.edu}

\begin{document}

\maketitle

\begin{abstract} A set of linearly independent eigenvectors, of a linear transformation,  can allow one to solve  problems in applied mathematics and in pure mathematics. We present a  condition on a set of eigenvectors that guarantees  linearly independence.

\end{abstract}

\section{Introduction}

We first review the relevant definitions and results that we will use in our proof. These topics are typically covered in a first course in linear algebra.

\begin{definition}\label{deflintrans} If  $T\colon V \to W$   is a function from a

vector space  $V$   to the vector space  $W,$  then  $T$   is

called a  linear transformation  if,  for all vectors  $\x$  and $\y$

in $V$    and for all scalars  $c,$   the following hold:

\bes

\item[(a)] $T(\x + \y) = T(\x) + T(\y)$

\item[(b)] $T(c\x) = cT(\x)$.

\ees

The vector space $V$  is called the domain of  $T$   and  the vector space $W$  is called the co-domain of $T.$

\end{definition}

\begin{theorem}\label{ltprop} If  $T\colon V \to W$  is a linear transformation, then

\bes

\item $T(\zero) = \zero$

\item $T(-\x) = -T(\x)$

\item $T(\x-\y) = T(\x)-T(\y)$

\item $T(a\x+b\y) = aT(\x)+bT(\y)$,

\ees

where  $a$ and $b$ are scalars, and $\zero$ denotes the zero vector in $V$ and $W$, respecitively

\end{theorem}

\begin{definition} Let $V$ be a vector space and let $T\colon V \to V$ be a linear transformation. A nonzero vector  $\v$  in $V$   is called an \emph{eigenvector} of $T$   if  $T(\v)$  is a scalar multiple of  $\v$, that is,

$T(\v) = \lambda\v$

for some scalar $\lambda$.   The scalar $\lambda$ is called an

{\it eigenvalue\/} of $T$, and the nonzero vector $\x$  is said to be an

{\it eigenvector corresponding} to the eigenvalue $\lambda$.

\end{definition}

\section{Eigenvectors Corresponding to Distinct Eigenvalues are Linearly Independent}

\begin{theorem}Let $T\colon V\to V$ be a linear transformation from a vector space $V$ to itself. Suppose that $T$ has  eigenvectors $\x\_1,\x\_2,\dots,\x\_k$ whose corresponding eigenvalues are all distinct. Then the vectors $\x\_1,\x\_2,\dots,\x\_k$ are linearly independent.

\end{theorem}

\begin{proof} Let $T\colon V \to V$ be a linear transformation. Suppose, for a contradiction, that the

 theorem is false for $T$. Thus, there must be a smallest number of eigenvectors $k$, for which the theorem is false. We will now work with this $k$. So $T$ has  eigenvectors $\x\_1,\x\_2,\dots,\x\_k$ and corresponding eigenvalues $\lambda\_1, \lambda\_2,\dots,\lambda\_k$ that are all distinct, that is,

$\lambda\_i\ne\lambda\_j$ when $i\ne j$ and $1\le i,j\le k$  such that $\x\_1,\x\_2,\dots,\x\_k$ are linearly {\bf de}pendent. First we observe that the eigenvector $\x\_1$ is linearly independent. To see this, suppose that $c\x\_1=\zero$. Since $\x\_1\ne \zero$ (because it is an eigenvector), it follows that $c=0$. Thus, $k>1$. Since $k$ is the smallest for which the theorem is false, it follows that the eigenvectors $\x\_1,\x\_2,\dots,\x\_{k-1}$ are linearly {\bf in}dependent.

Since the vectors $\x\_1,\x\_2,\dots,\x\_{k-1},\x\_k$ are linearly {\bf de}pendent, there are scalars \[c\_1, c\_2, \dots, c\_{k-1}, c\_k,\]which are {\bf not all $0$}, such that

\begin{equation}c\_1\x\_1+c\_2\x\_2+ \cdots + c\_{k-1}\x\_{k-1}+ c\_k\x\_k=\zero.\label{eigeqat}\end{equation}

We will now show that $c\_k\ne 0$. If $c\_k= 0$ then we would obtain, from \eqref{eigeqat}, the equation

\begin{equation}c\_1\x\_1+c\_2\x\_2+ \cdots + c\_{k-1}\x\_{k-1}=\zero.\label{eigeqatn}\end{equation}

Because $\x\_1,\x\_2,\dots,\x\_{k-1}$ are linearly {\bf in}dependent, we would be able to conclude from \eqref{eigeqatn} that all of the scalars $c\_1, c\_2, \dots, c\_{k-1}, c\_k$ must be $0$, which is not the case. Thus, we must have that  $c\_k\ne 0$.

Since $c\_k\ne 0$ and $\x\_k\ne\zero$, equation \eqref{eigeqat} implies that there is a  least one scalar $c\_\ell$ such that $c\_\ell\ne 0$ where $1\le \ell\le k-1$.

Applying $T$ to both sides of \eqref{eigeqat}, we get

\[c\_1T(\x\_1)+c\_2T(\x\_2)+ \cdots + c\_{k-1}T(\x\_{k-1})+ c\_kT(\x\_k)=T(\zero).\]

Since $T(\x\_i)=\lambda\x\_i$, for each $i=1,2,\dots k$, and $T(\zero)=\zero$ by Theorem~\ref{ltprop}(1), we conclude that

\begin{equation}c\_1\lambda\_1\x\_1+c\_2\lambda\_2\x\_2+ \cdots + c\_{k-1}\lambda\_{k-1}\x\_{k-1}+ c\_k\lambda\_k\x\_k=\zero.\label{eigeqat2}\end{equation}

Multiplying both sides of \eqref{eigeqat}  by the scalar $\lambda\_k$, we get

\begin{equation}c\_1\lambda\_k\x\_1+c\_2\lambda\_k\x\_2+ \cdots + c\_{k-1}\lambda\_k\x\_{k-1}+ c\_k\lambda\_k\x\_k=\zero.\label{eigeqat3}\end{equation}

Subtracting equation \eqref{eigeqat2} from equation \eqref{eigeqat3}, we see that

\[c\_1(\lambda\_k-\lambda\_1)\x\_1+c\_2(\lambda\_k-\lambda\_2)\x\_2+ \cdots + c\_{k-1}(\lambda\_k-\lambda\_{k-1})\x\_{k-1}=\zero.\]

Because $\x\_1,\x\_2,\dots,\x\_{k-1}$ are linearly independent, we have that

\[c\_1(\lambda\_k-\lambda\_1)=0, \s c\_2(\lambda\_k-\lambda\_2)=0, \s \dots, \s  c\_{k-1}(\lambda\_k-\lambda\_{k-1})=0.\]

Hence, in particular, $c\_\ell(\lambda\_k-\lambda\_\ell)=0$. We noted above that $c\_\ell\ne 0$. Thus $\lambda\_k-\lambda\_\ell=0$ and so, $\lambda\_k=\lambda\_\ell$. We conclude that not all of the given eigenvalues are distinct. This contradiction completes the proof of the theorem.

\end{proof}\end{document}