3.6 Cauchy Sequences

One of the problems with deciding if a sequence converges is that you need to have a purported limit before you can apply the limit definition. Augustin Cauchy found a way around this problem, called the Cauchy Convergence Criterion. A Cauchy sequence is a sequence whose elements become arbitrarily close to each other as the sequence progresses. Here is the formal definition.

Definition 3.6.1. A sequence $\langle s_n \rangle$ is called a **Cauchy Sequence** if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, if $m, n > N$, then $|s_n - s_m| < \varepsilon$.

Proof Strategy 3.6.2. To prove that a sequence $\langle s_n \rangle$ is Cauchy, we will use the proof diagram

Let $\varepsilon > 0$ be an arbitrary real number.
  Let $N =$ (the natural number you found).
    Let $m, n > N$ be arbitrary natural numbers.
      Prove $|s_n - s_m| < \varepsilon$.

Lemma 3.6.3. Every convergent sequence is a Cauchy sequence.

Proof. Let $\langle s_n \rangle$ be a convergent sequence. Let $\lim_{n\to\infty} s_n = s$. We shall prove that $\langle s_n \rangle$ is a Cauchy sequence. To do this, let $\varepsilon > 0$. Since $\lim_{n\to\infty} s_n = s$, there is an $N \in \mathbb{N}$ such that

$$\text{for all } n \in \mathbb{N} \text{ if } n > N \text{ then } |s_n - s| < \frac{\varepsilon}{2}. \tag{3.17}$$

Now let $m, n > N$. We shall prove that $|s_n - s_m| < \varepsilon$ as follows

$$\begin{aligned} |s_n - s_m| &= |(s_n - s) + (s - s_m)| && \text{by algebra.} \\ &= |s_n - s| + |s - s_m| && \text{by triangle inequality.} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && \text{by (3.17).} \\ &= \varepsilon && \text{by algebra.} \end{aligned}$$

Thus, the sequence $\langle s_n \rangle$ is a Cauchy sequence. This completes the proof of the lemma. □

Lemma 3.6.4. Every Cauchy sequence is bounded.

Proof. Let $\langle s_n \rangle$ be a Cauchy sequence. Thus, for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $|s_n - s_m| < \varepsilon$ for all $m, n > N$. So, lets take $\varepsilon = 1$ and let $N \in \mathbb{N}$ be so that $|s_n - s_m| < 1$ holds for all $m, n > N$. Thus,

$$|s_n| - |s_m| < |s_n - s_m| < 1$$

for all $m, n > N$. Let m_0 be any fixed natural number $m_0 > N$. Hence, $|s_n| < |s_{m_0}| + 1$ for all $n > N$. Let $M = \max\{|s_1|, \ldots, |s_N|, |s_{m_0}| + 1\}$. We see that $|s_n| \leq M$ for all $n \in \mathbb{N}$. Thus, we have that $-M \leq s_n \leq M$ for all $n \in \mathbb{N}$. Therefore, $\langle s_n \rangle$ is a bounded sequence. □

Theorem 3.6.5 (Cauchy Convergence Criterion)**.** Let $\langle s_n \rangle$ be a sequence. Then $\langle s_n \rangle$ is convergent if and only if $\langle s_n \rangle$ is a Cauchy sequence.

Proof. Let $\langle s_n \rangle$ be a sequence. We shall prove that $\langle s_n \rangle$ is convergent if and only if $\langle s_n \rangle$ is a Cauchy sequence.

($\Rightarrow$). Assume that $\langle s_n \rangle$ is convergent. Lemma 3.6.3 implies that $\langle s_n \rangle$ is a Cauchy sequence.

($\Leftarrow$). Assume that $\langle s_n \rangle$ is a Cauchy sequence. We shall prove that $\langle s_n \rangle$ is convergent. Lemma 3.6.4 implies that $\langle s_n \rangle$ is a bounded sequence. The Bolzano–Weierstrass Theorem 3.5.1 implies that $\langle s_n \rangle$

has a convergent subsequence $\langle s_{n_k}\rangle$. Let x be the limit of this subsequence $\langle s_{n_k}\rangle$. Using the fact that $\langle s_n\rangle$ is a Cauchy sequence, we can now prove that the sequence $\langle s_n\rangle$ also converges to x. To do this, let $\varepsilon > 0$. Since $\langle s_n\rangle$ is a Cauchy sequence, there is an $N \in \mathbb{N}$ such that

$$\text{for all } m, n \in \mathbb{N} \text{ if } m, n > N \text{ then } |s_n - s_m| < \frac{\varepsilon}{2}. \tag{3.18}$$

Now let $n > N$. We shall prove that $|s_n - x| < \varepsilon$. Because x is the limit of the subsequence $\langle s_{n_k}\rangle$, it follows that there is a natural number $n_k > N$ such that

$$|s_{n_k} - x| < \frac{\varepsilon}{2}. \tag{3.19}$$

Therefore,

$$\begin{aligned}
|s_n - x| &= |(s_n - s_{n_k}) + (s_{n_k} - x)| && \text{by algebra.}\\
&\le |s_n - s_{n_k}| + |s_{n_k} - x| && \text{by triangle inequality.}\\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && \text{by (3.18) and (3.19).}\\
&= \varepsilon && \text{by algebra.}
\end{aligned}$$

Thus, $|s_n - x| < \varepsilon$. This completes the proof of the theorem. □

The following lemma is a useful tool for showing that certain sequences are Cauchy.

Lemma 3.6.6. Let $\langle s_n\rangle$ be a sequence and let $a > 0$. Suppose for some r with $0 < r < 1$ we have

$$|s_{n+1} - s_n| \le ar^n \text{ for all } n \ge 1. \tag{3.20}$$

Then $\langle s_n\rangle$ is a Cauchy sequence and hence, converges.

Proof. Let $\langle s_n\rangle$, a, and r be as stated. By Corollary 3.1.15, we see that $\lim_{n\to\infty} ar^n = a \lim_{n\to\infty} r^n = 0$. By Theorem 3.6.5, the sequence $\langle ar^n\rangle$ is a Cauchy sequence. Now, let $\varepsilon > 0$. Because $\langle ar^n\rangle$ is a Cauchy sequence, there is a natural number N such that $(\star)$ $|ar^n - ar^m| < \varepsilon(1 - r)$ for all natural numbers $m, n > N$. Now, let $m, n > N$. We can assume that $m > n$. We show that $|s_m - s_n| < \varepsilon$ as follows:

$$\begin{aligned}
|s_m - s_n| &= |(s_{n+1} - s_n) + (s_{n+2} - s_{n+1}) + \cdots + (s_m - s_{m-1})| && \text{by algebra}^2\\
&\le |s_{n+1} - s_n| + |s_{n+2} - s_{n+1}| + \cdots + |s_m - s_{m-1}| && \text{by triangle inequality}\\
&\le ar^n + ar^{n+1} + \cdots + ar^{m-1} && \text{by (3.20)}\\
&= ar^n(1 + r + r^2 + \cdots + r^{m-n-1}) && \text{by algebra}\\
&= ar^n\left(\frac{1 - r^{m-n}}{1 - r}\right) && \text{by Theorem 1.4.5}\\
&= \frac{ar^n - ar^m}{1 - r} && \text{by algebra}\\
&< \frac{\varepsilon(1 - r)}{1 - r} = \varepsilon && \text{by } (\star) \text{ and algebra.}
\end{aligned}$$

□

[2] For example, $s_7 - s_3 = (s_4 - s_3) + (s_5 - s_4) + (s_6 - s_5) + (s_7 - s_6)$.

Exercises 3.6

1. Using Definition 3.6.1, prove that the sequence $\left\langle \frac{n}{n+3} \right\rangle$ is Cauchy.
2. Prove that any subsequence of a Cauchy sequence is also Cauchy.
3. Let $\langle s_n \rangle$ be a Cauchy sequence. Suppose that a subsequence $\langle s_{n_k} \rangle$ converges to ℓ. Prove that $\langle s_n \rangle$ also converges to ℓ.
4. Let $\langle s_n \rangle$ be a Cauchy sequence and let $k > 0$. Suppose that $\langle t_n \rangle$ is a sequence satisfying $|t_n - t_m| \leq k\,|s_n - s_m|$ for all $n, m \geq 1$. Prove that $\langle t_n \rangle$ is a Cauchy sequence.
5. Suppose that a sequence $\langle s_n \rangle$ satisfies $|s_{n+1} - s_n| \leq \frac{1}{n+1}$ for all $n \geq 1$. Must the sequence be Cauchy?
6. Suppose that the sequence $\langle s_n \rangle$ is such that $|s_n - s_m| \leq \frac{1}{mn}$ for all $m, n \in \mathbb{N}$.
   (a) Prove that $\langle s_n \rangle$ is a Cauchy sequence.
   (b) Prove that $\langle s_n \rangle$ is a constant sequence.
7. Suppose that the sequence $\langle s_n \rangle$ satisfies $|s_{n+1} - s_n| \leq \frac{1}{(n+1)!}$ for all $n \geq 1$. Show that $\langle s_n \rangle$ is a Cauchy sequence.
8. Consider the sequence $\langle s_n \rangle$ where $s_n = \sum_{k=1}^{n} \frac{1}{k!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$. Using Exercise 7 show that the sequence $\langle s_n \rangle$ converges.
9. Let $\langle s_n \rangle$ be a sequence where $s_1 \neq s_2$. Let $0 < r < 1$ and suppose that

$$|s_{n+2} - s_{n+1}| \leq r\,|s_{n+1} - s_n| \text{ for all } n \geq 1.$$

   Prove the following:
   (a) $|s_{n+1} - s_n| \leq r^{n-1}\,|s_2 - s_1|$ for all $n \geq 1$, by induction.
   (b) $\langle s_n \rangle$ is a Cauchy sequence.
10. Inductively define the sequence $\langle s_n \rangle$ by $s_1 = c > 0$ and $s_{n+1} = \frac{1}{2+s_n}$ for all $n \geq 1$. Observe that $s_n > 0$ for all $n \geq 1$.
   (a) Using Exercise 9(a), prove $\langle s_n \rangle$ is a Cauchy sequence.
   (b) Evaluate $\lim_{n\to\infty} s_n$.

Exercise Notes: For Exercise 1, $\left|\frac{n}{n+3} - \frac{m}{m+3}\right| = \left|\frac{n}{n+3} - 1 + 1 - \frac{m}{m+3}\right| \leq \left|\frac{n}{n+3} - 1\right| + \left|1 - \frac{m}{m+3}\right|$. For Exercises 2 and 3, use Lemma 3.3.4. For Exercise 5, consider the sequence in Example 3.1.27 on page 55. For part (b) of Exercise 6, let ℓ be the limit of the sequence. Show that $s_n = \ell$ for all n, by first showing that $|s_n - \ell| - |s_m - \ell| \leq |s_n - s_m|$ for all $m, n \in \mathbb{N}$. For Exercise 7, first show that $\frac{1}{(n+1)!} \leq \frac{1}{2^n}$ for all $n \geq 1$.

3.7 Infinite Limits

Some sequences "take off" in the positive or negative direction; that is, they increase or decrease without bound. Here is a precise definition of this notion.

Definition 3.7.1. Let $\langle s_n \rangle$ be a sequence.

- We say that $\langle s_n \rangle$ **diverges to** ∞ provided that for every $M > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n > N$, then $s_n > M$. In this case, we shall write $\lim_{n\to\infty} s_n = \infty$.