Now conclude that $\binom{m}{k}\frac{1}{m^k} < \frac{1}{k!} \leq \frac{1}{2^{k-1}}$.

7. Let $A \subseteq \mathbb{R}$ be nonempty and bounded. Let $\beta = \sup(A)$. Thus, for each $n \in \mathbb{N}$ there is an $b_n \in A$ such that $\beta - \frac{1}{n} < b_n$, and by Theorem 3.4.12 the sequence $\langle b_n \rangle$ has a monotone subsequence $\langle b_{n_k} \rangle$.

(a) Show that $\lim_{n\to\infty} b_n = \beta$.

(b) Prove that $\lim_{k\to\infty} b_{n_k} = \beta$.

(c) Suppose that $\beta \notin A$. Prove that $\langle b_{n_k} \rangle$ must be an increasing sequence.

8. Let $A \subseteq \mathbb{R}$ be nonempty and bounded. Let $\alpha = \inf(A)$. Thus, for each $n \in \mathbb{N}$ there is an $a_n \in A$ such that $a_n < \alpha + \frac{1}{n}$, and by Theorem 3.4.12 the sequence $\langle a_n \rangle$ has a monotone subsequence $\langle a_{n_k} \rangle$.

(a) Show that $\lim_{n\to\infty} a_n = \alpha$.

(b) Prove that $\lim_{k\to\infty} a_{n_k} = \alpha$.

(c) Suppose that $\alpha \notin A$. Prove that $\langle a_{n_k} \rangle$ must be a decreasing sequence.

3.5 Bolzano–Weierstrass Theorems

Bolzano–Weierstrass Theorem for sequences is a fundamental result about convergence which states that each bounded sequence in $\mathbb{R}$ has a convergent subsequence. This theorem is named after the mathematicians Bernard Bolzano and Karl Weierstrass. It was first proved by Bolzano, but his proof was lost. It was re-proven by Weierstrass and became an important centerpiece of analysis.

Theorem 3.5.1 (Bolzano–Weierstrass Theorem for sequences)**.** If the sequence $\langle s_n \rangle$ is bounded, then $\langle s_n \rangle$ has a convergent subsequence.

Proof. We are assuming that the sequence $\langle s_n \rangle$ is bounded. By the Monotone Subsequence Theorem 3.4.12, there is a monotone subsequence $\langle s_{n_k} \rangle$. Since $\langle s_n \rangle$ is bounded, it follows that $\langle s_{n_k} \rangle$ is bounded. Because $\langle s_{n_k} \rangle$ is a bounded monotone sequence, the Monotone Convergence Theorem 3.4.4 implies that $\langle s_{n_k} \rangle$ is a convergent subsequence. □

Definition 3.5.2. Let S be a subset of $\mathbb{R}$.

1. A point $x \in \mathbb{R}$ is an **accumulation point** of S if every neighborhood of x contains an infinite number of points from S. That is, if U is any neighborhood of x, then $S \cap U$ is infinite.
2. A point $x \in \mathbb{R}$ is an **isolated point of** S if $x \in S$ and x is not an accumulation point of S.

A point $x \in \mathbb{R}$ is an *accumulation point* of a set S if there are always an infinite number of points from the set S that are "very close" to x; that is, in every neighborhood of x. Thus, if $x \in I$ and I is an interval, then x is an an accumulation point of I.

A point x is an an *isolated point* of S if there is a neighborhood of x in which there are no other points from the set S (x is all alone; that is, x is the only point from S living in this neighborhood).

Remark 3.5.3. An accumulation point of S may be in the set S or may not be in S. On the other hand, an an isolated point must be in S.

Problem 3.5.4. For each of the following subsets S of $\mathbb{R}$ find some accumulation points (if any) and find some isolated points.

1. $S = [0, 3)$.

2. $S = \mathbb{N}$.
3. $S = \{\frac{1}{n} : n \in \mathbb{N}\}$.
4. $S = \{q \in \mathbb{Q} : 0 < q < 1\}$.

Theorem 3.5.5 (Bolzano–Weierstrass Theorem for sets)**.** Let $S \subseteq \mathbb{R}$ be infinite. If S is bounded, then there is a point $x \in \mathbb{R}$ such that x is an accumulation point of S.

Proof. Since S is infinite, there is a sequence $\langle s_n \rangle$ of distinct points from the set S, that is, $s_n \in S$ for all $n \in \mathbb{N}$. Since S is bounded, it follows that the sequence $\langle s_n \rangle$ is bounded. Theorem 3.5.1 implies that $\langle s_n \rangle$ has a convergent subsequence $\langle s_{n_k} \rangle$. Let x be the limit of this subsequence $\langle s_{n_k} \rangle$. We shall now prove that x is an accumulation point of S. That is, we shall prove that every neighborhood of x contains an infinite number of points from S. Corollary 3.1.20 implies that every neighborhood of x contains an infinite number of points from the subsequence $\langle s_{n_k} \rangle$. Since each $s_{n_k} \in S$, it follows that x is an accumulation point of S. □

Theorem 3.5.6. Let S be a nonempty set of real numbers and suppose that x is an accumulation point of S. Then there is a sequence of distinct points $\langle s_n \rangle$ in S that converges to x.

Proof. Observe that for any $n \geq 1$, a real number a is in the neighborhood $(x - \frac{1}{n}, x + \frac{1}{n})$ if and only if $|a - x| < \frac{1}{n}$. Now, for each $n \in \mathbb{N}$ we have that the $(x - \frac{1}{n}, x + \frac{1}{n})$ contains an infinite number of points in S. We now define a sequence with distinct points as follows: For $n = 1$ choose $s_1 \in S$ such that $|s_1 - x| < \frac{1}{1}$. For $n = 2$ choose an $s_2 \in S$ so that $s_2 \neq s_1$ and $|s_2 - x| < \frac{1}{2}$. For $n = 3$ choose an $s_3 \in S$ so that $s_3 \neq s_1, s_2$ and $|s_3 - x| < \frac{1}{3}$. Continuing in this manner, we obtain a sequence $\langle s_n \rangle$ of distinct points in S such that $|s_n - x| < \frac{1}{n}$ for all $n \geq 1$. Theorem 3.1.14 implies that the sequence $\langle s_n \rangle$ converges to x. □

Proposition 3.5.7. Let $\langle s_n \rangle$ be a sequence that does not converge to the real number ℓ. Then there is an $\varepsilon > 0$ and a subsequence $\langle s_{n_k} \rangle$ such that $|s_{n_k} - \ell| \geq \varepsilon$ for all $k \geq 1$. Hence, no subsequence of $\langle s_{n_k} \rangle$ can converge to ℓ.

Proof. Suppose $\langle s_n \rangle$ does not converge to ℓ. Thus there is an $\varepsilon > 0$ such that for all $N \in \mathbb{N}$ there is an $n > N$ such that $|s_n - \ell| \geq \varepsilon$ (see Remark 3.1.17). For $N = 1$ let $n_1 \geq 1$ be such that $|s_{n_1} - \ell| \geq \varepsilon$; for $N = n_1$, let $n_2 > n_1$ be such that $|s_{n_2} - \ell| \geq \varepsilon$; continuing in this manner, we obtain a subsequence $\langle s_{n_k} \rangle$ such that $|s_{n_k} - \ell| \geq \varepsilon$ for all $k \geq 1$. It follows that no subsequence of $\langle s_{n_k} \rangle$ can converge to ℓ (see Exercise 11 on page 64). □

Exercises 3.5

1. Can you find a sequence $\langle s_n \rangle$ such that $1 \leq s_n \leq 5$ for all $n \geq 1$, and $\langle s_n \rangle$ has no convergent subsequence?

2. Let $\langle s_n \rangle$ be a bounded sequence that does not converge. By Theorem 3.5.1 there is a subsequence $\langle s_{n_i} \rangle$ that converges to some real number ℓ. Show that there is another subsequence of $\langle s_n \rangle$ that converges to a real number different than ℓ.

3. Let $\mathbb{I} \subseteq \mathbb{R}$ be the set of irrational numbers. Find the set of all accumulation points of $\mathbb{I}$.

4. Let $S \subseteq \mathbb{R}$. Suppose that $\langle s_n \rangle$ is a sequence of distinct points in S that converges to x. Show that x is an accumulation point of S.

5. Let $\langle s_n \rangle$ be a sequence satisfying $|s_n - s_m| < M$ for all $n, m \geq 1$, where $M > 0$. Prove that $\langle s_n \rangle$ has a convergent subsequence.

Exercise Notes: For Exercise 2, Proposition 3.5.7 implies that there is a subsequence $\langle s_{n_k} \rangle$ that does not converge to ℓ. This subsequence is bounded.