

14. Suppose that  $\lim_{n \rightarrow \infty} s_n = \ell$  and  $|s_n| \leq M$  for all  $n \geq 1$ , where  $M > 0$ . Prove that  $\lim_{n \rightarrow \infty} s_n^2 = \ell^2$ .
15. Let  $c \in \mathbb{R}$  be constant. Prove that  $\lim_{n \rightarrow \infty} c = c$ .
16. Let  $\langle x_n \rangle$  and  $\langle y_n \rangle$  be two convergent sequences. Prove that there exists an  $M > 0$  such that  $|x_n| \leq M$  and  $|y_n| \leq M$  for all  $n \geq 1$ .
17. Let  $\langle s_n \rangle$  a convergent sequence. Suppose that  $\lim_{n \rightarrow \infty} s_n = \ell$ . Prove that there exists an  $M > 0$  such that  $|s_n + \ell| \leq M$  for all  $n \geq 1$ .
18. Use Theorems 3.1.14 and 3.1.5 to prove that  $\lim_{n \rightarrow \infty} \frac{\sin(n)}{n} = 0$ .
19. Use Theorem 3.1.14 and Theorem 3.1.6 to prove that  $\lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) = 0$ .
20. Let  $\langle s_n \rangle$  a convergent sequence. Suppose that  $\lim_{n \rightarrow \infty} s_n = \ell$ . Prove that  $\lim_{n \rightarrow \infty} s_n^2 = \ell^2$ .
21. Let  $x$  be a such that  $|x| > 1$ . Prove that the sequence  $\langle x^n \rangle$  is not bounded.

Exercise Notes: For Exercise 6, observe that  $n \leq 3n - 2$  when  $n \geq 1$ . For Exercise 7, observe that  $n < |2n - 7|$  when  $n > 7$ . In this case, we would need  $N$  to be at least 7 and so,  $N > \max\{7, \frac{14}{\varepsilon}\}$  could be used in the proof. For Exercise 19, show that  $|\sqrt{n+1} - \sqrt{n}| = \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{2\sqrt{n}}$ . For Exercise 21, see the proof of Corollary 3.1.15.

## 3.2 Limit Theorems for Sequences

What are Limit Theorems? A Limit Theorem states that if you know the limits of some given sequences, then you can determine the limit of a new sequence that is related to the given sequences. Limit theorems have the form:

**Theorem.** Suppose we are given that  $\lim_{n \rightarrow \infty} s_n = s$  and  $\lim_{n \rightarrow \infty} t_n = t$ . Then one can evaluate the limit of a new sequence, say  $\lim_{n \rightarrow \infty} u_n = u$ , which is constructed from the given sequences.

How does one prove theorems that have this form? We are assuming that  $\lim_{n \rightarrow \infty} s_n = s$  and  $\lim_{n \rightarrow \infty} t_n = t$ , and we must prove that  $\lim_{n \rightarrow \infty} u_n = u$ .

**Proof Diagram 3.2.1.** To prove that  $\lim_{n \rightarrow \infty} u_n = u$ , our proof must contain the structure

Assume  $\lim_{n \rightarrow \infty} s_n = s$ .

Assume  $\lim_{n \rightarrow \infty} t_n = t$ .

Let  $\varepsilon > 0$  be an arbitrary real number.

Let  $N =$  (the natural number you found).

Let  $n > N$  be an arbitrary natural number.

Prove  $|u_n - u| < \varepsilon$ .

Let  $\varepsilon > 0$  be given. We must find a natural number  $N$  such that if  $n > N$  then  $|u_n - u| < \varepsilon$ . We can use the assumptions  $\lim_{n \rightarrow \infty} s_n = s$  and  $\lim_{n \rightarrow \infty} t_n = t$ , to find the desired  $N$ . Here is the basic idea that we will apply to get  $N$ .

*Using algebra and properties of inequality on the expression  $|u_n - u|$ , we “extract out” a larger value containing  $|s_n - s|$  and  $|t_n - t|$ , and no other occurrences of  $s_n$  or  $t_n$ .*

Since  $\lim_{n \rightarrow \infty} s_n = s$  and  $\lim_{n \rightarrow \infty} t_n = t$ , we can make  $|s_n - s|$  and  $|t_n - t|$  “as small as we want.” We should then be able to make  $|u_n - u| < \varepsilon$  and find the  $N$  that we need. We will apply these ideas in our proof analysis of the next two theorems.

**Theorem 3.2.2.** Suppose  $\lim_{n \rightarrow \infty} s_n = s$  and  $\lim_{n \rightarrow \infty} t_n = t$ . Then  $\lim_{n \rightarrow \infty} (s_n + t_n) = s + t$ .

*Proof Analysis.* We are assuming that  $\lim_{n \rightarrow \infty} s_n = s$  and  $\lim_{n \rightarrow \infty} t_n = t$ . So we can make  $|s_n - s|$  and  $|t_n - t|$  as small as we want. We need to make

$$|(s_n + t_n) - (s + t)| < \varepsilon.$$

Using algebra and properties of inequality on the expression  $|(s_n + t_n) - (s + t)|$  we extract out  $|s_n - s|$  and  $|t_n - t|$  as follows:

$$\begin{aligned} |(s_n + t_n) - (s + t)| &= |(s_n - s) + (t_n - t)| && \text{by algebra.} \\ &\leq |s_n - s| + |t_n - t| && \text{by triangle inequality.} \end{aligned}$$

Thus, if we have that  $|s_n - s| < \frac{\varepsilon}{2}$  and  $|t_n - t| < \frac{\varepsilon}{2}$ , we can conclude that  $|(s_n + t_n) - (s + t)| < \varepsilon$ . Since  $\lim_{n \rightarrow \infty} s_n = s$ , we know that there is a natural number  $N_s$  such that  $|s_n - s| < \frac{\varepsilon}{2}$  when  $n > N_s$ . Similarly, we know that there is a natural number  $N_t$  such that  $|t_n - t| < \frac{\varepsilon}{2}$  when  $n > N_t$ . Thus, we let  $N = \max\{N_s, N_t\}$ . This will ensure that when  $n > N$  we will have that  $n > N_s$  and  $n > N_t$ . We can now compose a logically correct proof using proof strategy 3.2.1 as a guide.

*Proof.* Suppose  $\lim_{n \rightarrow \infty} s_n = s$  and  $\lim_{n \rightarrow \infty} t_n = t$ . We shall prove that  $\lim_{n \rightarrow \infty} (s_n + t_n) = s + t$ . Let  $\varepsilon > 0$ . Since  $\lim_{n \rightarrow \infty} s_n = s$ , there is an  $N_s \in \mathbb{N}$  such that

$$|s_n - s| < \frac{\varepsilon}{2} \text{ for all } n > N_s. \quad (3.6)$$

Also, since  $\lim_{n \rightarrow \infty} t_n = t$ , there is an  $N_t \in \mathbb{N}$  such that

$$|t_n - t| < \frac{\varepsilon}{2} \text{ for all } n > N_t. \quad (3.7)$$

Let  $N = \max\{N_s, N_t\}$ . Thus for  $n > N$  we have that

$$\begin{aligned} |(s_n + t_n) - (s + t)| &= |(s_n - s) + (t_n - t)| && \text{by algebra.} \\ &\leq |s_n - s| + |t_n - t| && \text{by triangle inequality.} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && \text{by (3.6) and (3.7).} \\ &= \varepsilon && \text{by algebra.} \end{aligned}$$

and therefore,  $|(s_n + t_n) - (s + t)| < \varepsilon$  for all  $n > N$ . This completes the proof of the theorem.  $\square$

In the proof of Theorem 3.2.2 we were able to ‘cleanly’ extract out  $|s_n - s|$  and  $|t_n - t|$ , that is, there were no additional factors. This may not be the case when applying strategy 3.2.1 to prove other such theorems. There may be times, after extracting out  $|s_n - s|$  or  $|t_n - t|$ , that we have some ‘unwanted’ factors involving  $n$ ,  $s_n$ , or  $t_n$ . If such factors appear, then you will have to find an appropriate upper bound for these factors. This will be done in the proofs of Theorems 3.2.3 and 3.2.6. In our proof analysis of Theorem 3.2.3 we obtain the unwanted factor  $|t_n|$ . As the sequence  $\langle t_n \rangle$  converges, there is a  $K > 0$  such that  $|t_n| \leq K$  for all  $n \geq 1$ . So we will have a desired upper bound. Actually, in our proof analysis, we use  $M = \max\{K, |s|\}$  for the upper bound.

**Theorem 3.2.3.** Suppose  $\lim_{n \rightarrow \infty} s_n = s$  and  $\lim_{n \rightarrow \infty} t_n = t$ . Then  $\lim_{n \rightarrow \infty} (s_n t_n) = st$ .

*Proof Analysis.* We are assuming that  $\lim_{n \rightarrow \infty} s_n = s$  and  $\lim_{n \rightarrow \infty} t_n = t$ . So we can make  $|s_n - s|$  and  $|t_n - t|$  as small as we want. We need to make  $|s_n t_n - st| < \varepsilon$ . Since the sequence  $\langle t_n \rangle$  converge, we know by Theorem 3.1.25 that this sequence is bounded. Thus, there is an  $M > 0$  such that

$$|s| \leq M \text{ and } |t_n| \leq M \text{ for all } n \geq 1. \quad (3.8)$$

We can now use algebra and properties of inequality on the expression  $|s_n t_n - st|$  to extract out  $|s_n - s|$  and  $|t_n - t|$  as follows:

$$\begin{aligned} |s_n t_n - st| &= |s_n t_n - st_n + st_n - st| && \text{by algebra.}^1 \\ &= |t_n(s_n - s) + s(t_n - t)| && \text{by algebra.} \\ &\leq |t_n(s_n - s)| + |s(t_n - t)| && \text{by triangle inequality.} \\ &= |t_n| |s_n - s| + |s| |t_n - t| && \text{by property of } | \cdot |. \\ &\leq M |s_n - s| + M |t_n - t| && \text{by (3.8).} \end{aligned}$$

Thus, if we have that  $|s_n - s| < \frac{\varepsilon}{2M}$  and  $|t_n - t| < \frac{\varepsilon}{2M}$ , we can conclude that  $|s_n t_n - st| < \varepsilon$ . We can now compose a logically correct proof using proof strategy 3.2.1 as a guide.

*Proof.* See Exercise 8. □

Our next lemma shows that if the limit of a sequence is nonzero then the terms of the sequence are eventually nonzero as well.

**Lemma 3.2.4.** Suppose  $\lim_{n \rightarrow \infty} t_n = t$  and that  $t \neq 0$ . Then there is a real number  $\delta > 0$  and  $N \in \mathbb{N}$  such that  $\delta < |t|$  and  $\delta < |t_n|$  for all  $n > N$ .

*Proof.* Since  $|t| > 0$ , let  $\delta = \frac{|t|}{2}$ . Clearly,  $0 < \delta < |t|$ . Now, because  $\lim_{n \rightarrow \infty} t_n = t$ , there is an  $N \in \mathbb{N}$  such that  $|t - t_n| < \delta$  for all  $n > N$ . Thus for  $n > N$ , since  $|t| - |t_n| \leq |t - t_n|$ , we have that  $|t| - |t_n| < \delta$ . Hence,  $|t| - \delta < |t_n|$  and so,  $|t| - \frac{|t|}{2} < |t_n|$ . Since  $|t| - \frac{|t|}{2} = \frac{|t|}{2} = \delta$ , we have that  $\delta < |t_n|$  for all  $n > N$ . □

The following lemma shows that if all the terms of a convergent sequence are nonzero and the limit is also nonzero, then there is a single positive number  $\gamma$  smaller than the absolute values of the limit and all these terms.

**Lemma 3.2.5.** Suppose  $\lim_{n \rightarrow \infty} t_n = t$  where  $t \neq 0$  and  $t_n \neq 0$  for all  $n \geq 1$ . Then there is a real number  $\gamma > 0$  such that  $\gamma \leq |t|$  and  $\gamma \leq |t_n|$  for all  $n \geq 1$ .

*Proof.* Suppose  $\lim_{n \rightarrow \infty} t_n = t$ ,  $t_n \neq 0$  for all  $n \in \mathbb{N}$  and  $t \neq 0$ . By Lemma 3.2.4 there is a  $\delta > 0$  and an  $N \in \mathbb{N}$  such that  $\delta < |t|$  and  $\delta < |t_n|$  for all  $n > N$ . Let  $\gamma = \min\{|t_1|, |t_2|, \dots, |t_N|, \delta\}$ . Clearly,  $\gamma > 0$  and  $\gamma \leq |t|$  and  $\gamma \leq |t_n|$  for all  $n \geq 1$ . □

Our next theorem shows that if a convergent sequence  $\langle t_n \rangle$  satisfies the conditions of Lemma 3.2.5, then the reciprocal sequence  $\langle \frac{1}{t_n} \rangle$  also converges.

**Theorem 3.2.6.** Suppose  $t \neq 0$  and  $t_n \neq 0$  for all  $n \geq 1$ . If  $\lim_{n \rightarrow \infty} t_n = t$ , then  $\lim_{n \rightarrow \infty} \frac{1}{t_n} = \frac{1}{t}$ .

*Proof Analysis.* We are assuming that  $\lim_{n \rightarrow \infty} t_n = t$ . So we can make  $|t_n - t|$  as small as we want. We need to make  $|\frac{1}{t_n} - \frac{1}{t}| < \varepsilon$ . Since  $t \neq 0$  and  $t_n \neq 0$  for all  $n \geq 1$ , we know by Lemma 3.2.5

<sup>1</sup>This algebraic “trick,” of adding and subtracting the same value, is used often in analysis.

that there is a real number  $\gamma > 0$  such that  $(\star)$   $\gamma \leq |t|$  and  $\gamma \leq |t_n|$  for all  $n \geq 1$ . We can now use algebra and properties of inequality on the expression  $\left| \frac{1}{t_n} - \frac{1}{t} \right|$  to extract out  $|t_n - t|$  as follows:

$$\begin{aligned} \left| \frac{1}{t_n} - \frac{1}{t} \right| &= \left| \frac{t - t_n}{t_n t} \right| && \text{by algebra.} \\ &= \frac{|t - t_n|}{|t_n| |t|} && \text{by property of abs. val.} \\ &\leq \frac{|t - t_n|}{\gamma^2} && \text{by } (\star). \end{aligned}$$

Thus, if we have that  $|t_n - t| < \gamma^2 \varepsilon$ , we can conclude that  $\left| \frac{1}{t_n} - \frac{1}{t} \right| < \varepsilon$ . We can now compose a logically correct proof using proof strategy 3.2.1 as a guide.

*Proof.* Assume that  $t \neq 0$ ,  $t_n \neq 0$  for all  $n \geq 1$ , and  $\lim_{n \rightarrow \infty} t_n = t$ . By Lemma 3.2.5, there is a real number  $\gamma > 0$  such that  $\gamma \leq |t|$  and  $\gamma \leq |t_n|$  for all  $n \geq 1$ . Thus,

$$\gamma^2 \leq |t_n t| \text{ for all } n \geq 1. \quad (3.9)$$

To prove that  $\lim_{n \rightarrow \infty} \frac{1}{t_n} = \frac{1}{t}$ , let  $\varepsilon > 0$ . Since  $\lim_{n \rightarrow \infty} t_n = t$ , there is an  $N \in \mathbb{N}$  such that

$$|t_n - t| < \varepsilon \gamma^2 \text{ for all } n > N. \quad (3.10)$$

Thus for  $n > N$  we have that

$$\begin{aligned} \left| \frac{1}{t_n} - \frac{1}{t} \right| &= \left| \frac{t - t_n}{t_n t} \right| && \text{by algebra.} \\ &= \frac{|t - t_n|}{|t_n| |t|} && \text{by property of abs. val.} \\ &\leq \frac{|t - t_n|}{\gamma^2} && \text{by (3.9).} \\ &< \frac{\varepsilon \gamma^2}{\gamma^2} && \text{by (3.10).} \\ &= \varepsilon && \text{by algebra.} \end{aligned}$$

Therefore,  $\left| \frac{1}{t_n} - \frac{1}{t} \right| < \varepsilon$  for all  $n > N$ . This completes the proof of the theorem.  $\square$

**Theorem 3.2.7.** Suppose that  $\lim_{n \rightarrow \infty} s_n = s$  and  $\lim_{n \rightarrow \infty} t_n = t$ . If  $t \neq 0$  and  $t_n \neq 0$  for all  $n \geq 1$ , then  $\lim_{n \rightarrow \infty} \frac{s_n}{t_n} = \frac{s}{t}$ .

*Proof.* Suppose that  $\lim_{n \rightarrow \infty} s_n = s$ ,  $\lim_{n \rightarrow \infty} t_n = t$ ,  $t \neq 0$  and  $t_n \neq 0$  for all  $n \geq 1$ . Since  $\frac{s_n}{t_n} = s_n \left( \frac{1}{t_n} \right)$ , Theorems 3.2.6 and 3.2.3 imply the desired conclusion.  $\square$

**Theorem 3.2.8** (Squeeze Theorem). Let  $\langle s_n \rangle$ ,  $\langle y_n \rangle$  and  $\langle t_n \rangle$  be sequences where  $\lim_{n \rightarrow \infty} s_n = a$  and  $\lim_{n \rightarrow \infty} t_n = a$ . Suppose, for some  $m \in \mathbb{N}$ , that  $s_n \leq y_n \leq t_n$  for all  $n \geq m$ . Then  $\lim_{n \rightarrow \infty} y_n = a$ .

*Proof.* Let  $\langle s_n \rangle$ ,  $\langle y_n \rangle$ ,  $\langle t_n \rangle$ ,  $m$  be as stated. Let  $\varepsilon > 0$ . Since  $\lim_{n \rightarrow \infty} s_n = a$ , there is an  $N_s \in \mathbb{N}$  such that

$$|s_n - a| < \varepsilon \text{ for all } n > N_s. \quad (3.11)$$

Since  $\lim_{n \rightarrow \infty} t_n = a$ , there is an  $N_t \in \mathbb{N}$  such that

$$|t_n - a| < \varepsilon \text{ for all } n > N_t. \quad (3.12)$$

Let  $N = \max\{m, N_s, N_t\}$ . Let  $n > N$ . We shall prove that  $|y_n - a| < \varepsilon$ . By assumption, we have that  $s_n \leq y_n \leq t_n$  and so,  $s_n - a \leq y_n - a \leq t_n - a$ . By Lemma 2.1.7, we have that  $|y_n - a| \leq \max\{|s_n - a|, |t_n - a|\}$ . Since  $n > N_s$  and  $n > N_t$ , we conclude from (3.11) and (3.12) that  $\max\{|s_n - a|, |t_n - a|\} < \varepsilon$ . Hence,  $|y_n - a| < \varepsilon$ .  $\square$

**Corollary 3.2.9.** Let  $c$  be a such that  $c > 0$ . Then  $\lim_{n \rightarrow \infty} c^{\frac{1}{n}} = 1$ .

*Proof.* Let  $c$  be a such that  $0 < c$ . Suppose that  $c \geq 1$ . Let  $n \geq 1$  be a natural number. Let  $d \geq 0$  be such that  $1 + d = c$ . Since  $d \geq 0$ , we see that  $\frac{d}{n} > -1$ . By Bernoulli's inequality (see Exercise 5 on page 28), we have that

$$\left(1 + \frac{d}{n}\right)^n \geq 1 + n \frac{d}{n} = 1 + d = c \geq 1.$$

Hence,  $(1 + \frac{d}{n}) \geq c^{\frac{1}{n}} \geq 1$ , for all  $n \geq 1$ . Since  $\lim_{n \rightarrow \infty} (1 + \frac{d}{n}) = 1$  and  $\lim_{n \rightarrow \infty} 1 = 1$ , Theorem 3.2.8 now implies that  $\lim_{n \rightarrow \infty} c^{\frac{1}{n}} = 1$ . If  $0 < c < 1$ , then  $\frac{1}{c} > 1$ . Thus, the above argument shows that  $\lim_{n \rightarrow \infty} \frac{1}{c^{\frac{1}{n}}} = 1$ . Theorem 3.2.6 now implies that  $\lim_{n \rightarrow \infty} c^{\frac{1}{n}} = 1$ .  $\square$

**Corollary 3.2.10.**  $\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$ .

*Proof.* Define the sequence  $\langle b_n \rangle$  by  $b_n = n^{\frac{1}{n}} - 1$ , for all  $n \in \mathbb{N}$ . It is sufficient to show that  $\langle b_n \rangle$  converges to 0. Since  $n^{\frac{1}{n}} \geq 1$ , we see that  $b_n \geq 0$ , for each  $n \in \mathbb{N}$ . Let  $n \in \mathbb{N}$ . Since  $n^{\frac{1}{n}} = b_n + 1$ , we have  $n = (1 + b_n)^n$ . Thus, by Corollary 1.4.12,

$$n = (1 + b_n)^n > \frac{n(n-1)}{2} b_n^2$$

when  $n \geq 2$ . Thus,  $\frac{2}{n-1} > b_n^2$  and  $\frac{\sqrt{2}}{\sqrt{n-1}} > b_n \geq 0$  for all  $n \geq 2$ . Since  $\lim_{n \rightarrow \infty} \frac{\sqrt{2}}{\sqrt{n-1}} = 0$  and  $\lim_{n \rightarrow \infty} 0 = 0$ , Theorem 3.2.6 now implies that  $\lim_{n \rightarrow \infty} b_n = 0$ . Therefore,  $\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$ .  $\square$

We now present another “squeeze” result that will be used later in the book.

**Lemma 3.2.11.** Suppose that  $\lim_{n \rightarrow \infty} x_n = c$ . If a sequence  $\langle y_n \rangle$  satisfies  $|x_n - y_n| < \frac{1}{n}$  for all  $n \geq 1$ , then  $\lim_{n \rightarrow \infty} y_n = c$ .

*Proof.* See Exercise 4.  $\square$

### 3.2.1 Order Preserving Limit Theorems

**Theorem 3.2.12.** Suppose  $\lim_{n \rightarrow \infty} t_n = t$ . If  $t_n \geq 0$  for all  $n \in \mathbb{N}$ , then  $t \geq 0$ .

*Proof.* Suppose  $\lim_{n \rightarrow \infty} t_n = t$  and

$$t_n \geq 0 \text{ for all } n \in \mathbb{N}. \quad (3.13)$$

We shall prove that  $t \geq 0$ . Suppose, for a contradiction, that  $t < 0$ . Then  $-t > 0$  and let  $\varepsilon = \frac{-t}{2} > 0$ . Thus, there is an  $N \in \mathbb{N}$  such that  $|t_n - t| < \varepsilon$  for all  $n > N$ . So, let  $n > N$ . Thus, since  $t_n - t \leq |t_n - t| < \varepsilon$ , we conclude that  $t_n - t < \varepsilon = \frac{-t}{2}$ . Hence,  $t_n < t - \frac{t}{2} = \frac{t}{2}$ . Therefore,  $t_n < \frac{t}{2}$ . Now, since  $t < 0$  we see that  $\frac{t}{2} < 0$ . We conclude that  $t_n < 0$  which contradicts (3.13). This completes the proof of the theorem.  $\square$

**Theorem 3.2.13.** Suppose  $\lim_{n \rightarrow \infty} s_n = s$  and  $\lim_{n \rightarrow \infty} t_n = t$ . If  $s_n \leq t_n$  for all  $n \in \mathbb{N}$ , then  $s \leq t$ .

*Proof.* Suppose  $\lim_{n \rightarrow \infty} s_n = s$ ,  $\lim_{n \rightarrow \infty} t_n = t$  and that  $s_n \leq t_n$  for all  $n \in \mathbb{N}$ . Hence,  $0 \leq (t_n - s_n)$  for all  $n \in \mathbb{N}$ . Exercise 1 on page 61 implies that  $\lim_{n \rightarrow \infty} (t_n - s_n) = t - s$ . Theorem 3.2.12 now implies that  $0 \leq t - s$  and therefore,  $s \leq t$ .  $\square$

**Theorem 3.2.14.** Let  $\langle s_n \rangle$  be a sequence in the closed interval  $[a, b]$ ; that is,  $a \leq s_n \leq b$  for all  $n \in \mathbb{N}$ . If  $\lim_{n \rightarrow \infty} s_n = c$ , then  $c$  is in  $[a, b]$ .

*Proof.* Suppose  $\lim_{n \rightarrow \infty} s_n = c$  and that  $s_n \leq b$  for all  $n \in \mathbb{N}$ . We shall prove that  $c \leq b$ . Since the constant sequence  $\langle b \rangle$  converges to  $b$  and  $s_n \leq b$  for all  $n \in \mathbb{N}$ , Theorem 3.2.13 implies that  $c \leq b$ . We shall now prove that  $a \leq c$ . Since the constant sequence  $\langle a \rangle$  converges to  $a$  and  $a \leq s_n$  for all  $n \in \mathbb{N}$ , Theorem 3.2.13 implies that  $a \leq c$ . Therefore,  $a \leq c \leq b$  and hence  $c$  is in  $[a, b]$ .  $\square$

**Corollary 3.2.15.** Let  $\langle s_n \rangle$  be a sequence such that  $\lim_{n \rightarrow \infty} s_n = c$  and let  $a, b$  be real numbers.

1. If  $s_n \leq b$  for all  $n \in \mathbb{N}$ , then  $c \leq b$ .
2. If  $a \leq s_n$  for all  $n \in \mathbb{N}$ , then  $a \leq c$ .

*Proof.* This follows from the proof of Theorem 3.2.14.  $\square$

**Corollary 3.2.16.** Let  $\langle s_n \rangle$  be a sequence such that  $\lim_{n \rightarrow \infty} s_n = c$  and let  $M \geq 0$ . If  $|s_n| \leq M$  for all  $n \in \mathbb{N}$ , then  $|c| \leq M$ .

*Proof.* If  $|s_n| \leq M$  for all  $n \in \mathbb{N}$ , then  $-M \leq s_n \leq M$  for all  $n \in \mathbb{N}$ . Corollary 3.2.15 implies that  $-M \leq c \leq M$ , that is,  $|c| \leq M$ .  $\square$

### Exercises 3.2

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1. Suppose  $\lim_{n \rightarrow \infty} s_n = s$  and  $\lim_{n \rightarrow \infty} t_n = t$ . Using proof strategy 3.2.1 as a guide, prove that  $\lim_{n \rightarrow \infty} (s_n - t_n) = s - t$ .
2. Suppose  $\lim_{n \rightarrow \infty} s_n = s$  and  $\lim_{n \rightarrow \infty} t_n = t$ . Let  $a, b \in \mathbb{R}$  be nonzero. Using proof strategy 3.2.1 as a guide, prove that  $\lim_{n \rightarrow \infty} (as_n + bt_n) = as + bt$ .
3. Let  $\langle s_n \rangle$  and  $\langle t_n \rangle$  be sequences. Suppose that  $\langle s_n \rangle$  converges and  $\gamma > 0$  is such that  $\gamma \leq |t_n|$  for all  $n \geq 1$ . Prove that the sequence  $\left\langle \frac{s_n}{t_n} \right\rangle$  is bounded.
4. Prove Lemma 3.2.11. [Hint:  $|y_n - c| = |y_n - x_n + x_n - c|$ .]
5. Let  $\langle a_n \rangle$  and  $\langle b_n \rangle$  be sequences. Suppose that  $\lim_{n \rightarrow \infty} (a_n + b_n) = \ell$  and  $\lim_{n \rightarrow \infty} (a_n - b_n) = m$  where  $\ell, m \in \mathbb{R}$ . Use Theorem 3.2.2 to prove that  $\langle a_n \rangle$  converges and evaluate its limit.
6. Let  $\langle a_n \rangle$  and  $\langle b_n \rangle$  be sequences. Suppose that  $\lim_{n \rightarrow \infty} (a_n + b_n) = \ell$  and  $\lim_{n \rightarrow \infty} (a_n - b_n) = m$  where  $\ell, m \in \mathbb{R}$ . Use Exercise 1 to prove that  $\langle b_n \rangle$  converges and evaluate its limit.
7. Let  $\langle a_n \rangle$  be a bounded sequence and suppose that  $\lim_{n \rightarrow \infty} b_n = 0$ . Prove that  $\lim_{n \rightarrow \infty} a_n b_n = 0$ .
8. Prove Theorem 3.2.3.
9. Let  $k \in \mathbb{N}$ . Show that  $\lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^k = 1$ .
10. Prove that  $\lim_{n \rightarrow \infty} (n+1)^{\frac{1}{n}} = 1$ .
11. Prove that  $\lim_{n \rightarrow \infty} (n-1)^{\frac{1}{n}} = 1$ .

