So, $|x| \le |x - y| + |y|$ and thus, $|x| - |y| \le |x - y|$. Now we prove (iv). Observe that $|y| = |y - x + x|$. Hence, the triangle inequality implies that

$$|y| = |y - x + x| \le |y - x| + |x|.$$

Since $|y - x| = |x - y|$, we conclude that $|y| \le |x - y| + |x|$ and thus, $|y| - |x| \le |x - y|$. $\qquad \square$

Given a finite nonempty set of real numbers A, we let $\max A$, or $\max(A)$, denote the maximum number in A. Similarly, we define $\min A$, or $\min(A)$, to be the minimum number in A. For example, $\max\{-1, 2, \pi, 3\} = \pi$ and $\min\{-1, 2, \pi, 3\} = -1$.

Lemma 2.1.6. Let x, a and b be real numbers. If $a \le x \le b$, then $|x| \le \max\{|a|, |b|\}$.

Proof. Assume that $a \le x \le b$. We shall prove that $|x| \le \max\{|a|, |b|\}$. First suppose that $x \ge 0$. So, because $x \le b$, we conclude that $b \ge 0$. So, $|x| = x$ and $|b| = b$. Since $x \le b$, we see that $|x| \le |b|$ and therefore, $|x| \le \max\{|a|, |b|\}$. Now suppose that $x < 0$. Then, because $a \le x$, we see that $a < 0$. So, $|x| = -x$ and $|a| = -a$. Since $a \le x$, we see that $-x \le -a$ and so, $|x| \le |a|$. Therefore, $|x| \le \max\{|a|, |b|\}$. $\qquad \square$

Lemma 2.1.7. Let $a \le x \le b$ and $a \le y \le b$. Then $|x - y| \le |b - a|$.

Proof. Assume (1) $a \le x \le b$ and $a \le y \le b$. Thus, (2) $-b \le -y \le -a$. Adding (1) and (2), we obtain $-(b - a) \le x - y \le (b - a)$. Thus, $|x - y| \le |b - a|$ by Theorem 2.1.4(c). $\qquad \square$

Exercises 2.1

1. Prove Theorem 2.1.1.
2. Prove (c) of Theorem 2.1.4.
3. Prove (i)-(iii) of Theorem 2.1.5.
4. Justify the equalities and inequalities used in the proof of Theorem 2.1.4(f) on page 30.
5. Let $a > 0$. Prove that if $|x - a| < a$, then $x > 0$.
6. Suppose that $x > 0$ and $y > 0$. Prove that $x^2 + y^2 < (x + y)^2$.
7. Suppose that $x < 0$ and $y < 0$. Prove that $x^2 + y^2 < (x + y)^2$.
8. Suppose that $x > 0$ and $y > 0$. Prove that $\sqrt{x + y} < \sqrt{x} + \sqrt{y}$.
9. Let $a < x < b$ and $a < y < b$. Prove that $|x - y| < |b - a|$.

Exercise Notes: For Exercise 5, use Theorem 2.1.5(i). For Exercise 8 use proof by contradiction and Theorem 1.1.15.

2.2 The Completeness Axiom

In this section we present the completeness axiom, an assertion that implies there are no gaps or holes in the real number line. We shall see that this axiom is an essential property of the real number system that will allows us to conclude many of the fundamental theorems in the calculus; for example, the Cauchy Convergence Criterion, the Intermediate Value Theorem, and the Fundamental Theorem of Calculus.

Definition 2.2.1 (Upper Bound and Lower Bound)**.** Let $S \subseteq \mathbb{R}$ be nonempty.

- Suppose there is an $b \in \mathbb{R}$ such that every real number in S is less than or equal to b, that is, for all $x \in S$ we have $x \leq b$. Then we shall say that b is an **upper bound** for S and that S is **bounded above.**

- Suppose there is an $a \in \mathbb{R}$ such that a is less than or equal to every real number in S, that is, for all $x \in S$ we have $a \leq x$. Then we will say that a is a **lower bound** for S and that S is **bounded below.**

- If S has both a lower bound and an upper bound, then we say that S is **bounded**.

Let $S = \{\frac{2}{n} : n \in \mathbb{N}\}$ (see Figure 2.1). Since $\frac{2}{n} \leq 3$ for all $n \in \mathbb{N}$, we see that the set S is bounded above. Furthermore, because $0 \leq \frac{2}{n}$ for all $n \in \mathbb{N}$, the set S is bounded below.

Theorem 2.2.2. Let $S \subseteq \mathbb{R}$ be nonempty. Then S is bounded if and only if there is an $M > 0$ so that $|x| \leq M$ for all $x \in S$.

Proof. Let $S \subseteq \mathbb{R}$ be nonempty. Assume that S is bounded. Thus, there are nonzero real numbers a, b such that $a \leq x \leq b$ for all $x \in S$. Let $M = \max\{|a|, |b|\} > 0$. Lemma 2.1.6 implies that $|x| \leq M$ for all $x \in S$. To prove the converse, assume that $|x| \leq M$ for all $x \in S$ where $M > 0$. Thus, $-M \leq x \leq M$ for all $x \in S$. Thus, S is bounded. $\qquad\square$

Definition 2.2.3 (Least Upper Bound and Greatest Lower Bound)**.** Let $S \subseteq \mathbb{R}$ be nonempty.

- Suppose that β is an upper bound for S. If β is the **least upper bound** for S, then β is called the **supremum**[1] of S and we write $\beta = \sup(S)$.

- Suppose that α is a lower bound for S. If α is the **greatest lower bound** for S, then α is called the **infimum**[2] of S and we write $\alpha = \inf(S)$.

Example 2.2.4. Consider the set $S = \{\frac{2}{n} : n \in \mathbb{N}\}$ (see Figure 2.1). As we saw earlier, the set S is bounded. We note that 0 is the greatest lower bound for S and that 2 is the least upper bound for S. Thus, $\inf(S) = 0$ and $\sup(S) = 2$.
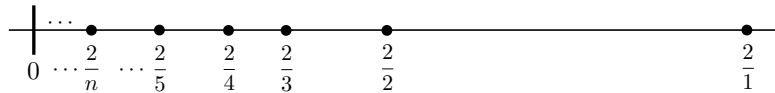
Figure 2.1: The set $\{\frac{2}{n} : n \in \mathbb{N}\}$ plotted on the real line

Let $S \subseteq \mathbb{R}$ be nonempty. The equation $\beta = \sup(S)$ means that the following two items hold: (a) β is an upper bound for S and (b) β is the smallest upper bound for S. Similarly, the equation $\alpha = \inf(S)$ means that (a) α is a lower bound for S and (b) α is the largest lower bound for S. The next remark repeats and clarifies these observations. This remark will be used to identify two important proof strategies.

Remark 2.2.5. Let $S \subseteq \mathbb{R}$ be nonempty and let $\alpha, \beta \in \mathbb{R}$. Then we have:

1. $\beta = \sup(S)$ if and only if the following two conditions hold:

   (a) For all $x \in S$, $x \leq \beta$.
   (b) For all real numbers b, if b is an upper bound for S, then $\beta \leq b$.

---

[1] Suprema is the plural form of the singular noun supremum.
[2] Infima is the plural form of the singular noun infimum.

2. $\alpha = \inf(S)$ if and only if the following two conditions hold:

    (a) For all $x \in S$, $\alpha \le x$.

    (b) For all real numbers a, if a is a lower bound for S, then $a \le \alpha$.
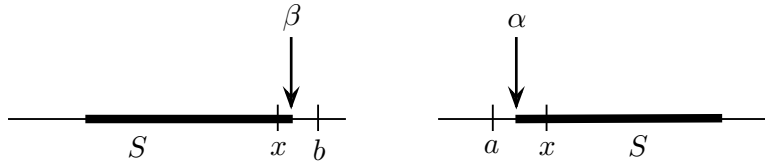

Figure 2.2: Illustrations for parts 1 and 2 of Remark 2.2.5, respectively

The following axiom captures one of the most important properties of the set of real numbers.

Completeness Axiom. Every nonempty $S \subseteq \mathbb{R}$ that is bounded above, has a least upper bound.

The completeness axiom just asserts that if a set $S \subseteq \mathbb{R}$ is nonempty and bounded above, then there is a real number β that satisfies the equation $\beta = \sup(S)$.

2.2.1 Proofs on the Supremum of a Set

Remark 2.2.5(1) inspires the following useful strategy for proving equations of the form $\sup(S) = \beta$.

Proof Strategy 2.2.6. Given a real number β and a nonempty $S \subseteq \mathbb{R}$, to prove that $\sup(S) = \beta$ we may use the two-step proof diagram

> Step (1): Prove $x \le \beta$ for all $x \in S$.
> Step (2): Assume b is an upper bound for S.
>   Prove $\beta \le b$.

In other words, to prove that $\sup(S) = \beta$ you must first prove that β is an upper bound for S and then prove that β is the smallest upper for S. Similarly, Remark 2.2.5(1) yields a strategy that allows one to take advantage of an assumption having the form $\sup(S) = \beta$.

Assumption Strategy 2.2.7. Let $S \subseteq \mathbb{R}$ be nonempty and let $\beta \in \mathbb{R}$. Suppose that you are *assuming* that $\sup(S) = \beta$. Then (1) $x \le \beta$ for all $x \in S$, and (2) whenever b is an upper bound for S, you can conclude that $\beta \le b$.

Our proof of the next lemma employs both proof strategy 2.2.6 and assumption strategy 2.2.7. Given a set $S \subset \mathbb{R}$ and a real number $k \in \mathbb{R}$ we can form the new set of real numbers defined by $A = \{kx : x \in S\}$. The set A is sometimes denoted by kS.

Lemma 2.2.8. Let $S \subseteq \mathbb{R}$ be nonempty and bounded above, and let $k \ge 0$. Define the set A by $A = \{kx : x \in S\}$. Then the set A is bounded above and $\sup(A) = k\sup(S)$.

Proof Analysis. Since the set S is bounded above there is a real number β satisfying $\beta = \sup(S)$ by the completeness axiom. So, we must prove that $\sup(A) = k\beta$. Since $k \ge 0$, there are two cases to consider: $k = 0$ and $k > 0$. When $k = 0$ the proof follows easily (see proof below). Let us suppose that $k > 0$. First note that every element in A has the form kx for $x \in S$. Thus, to prove that $k\beta$

is an upper bound for A, we just need to prove that $kx \leq k\beta$ for all $x \in S$. Appealing to proof strategy 2.2.6, we construct the following proof diagram:

$$\text{Assume } \beta = \sup(S).$$
$$\text{Prove } kx \leq k\beta \text{ for all } x \in S.$$
$$\text{Assume } c \text{ is an upper bound for } A.$$
$$\text{Prove } k\beta \leq c.$$

The first line of the above proof diagram indicates that, in our proof, we will be assuming $\beta = \sup(S)$. Thus, we can use assumption strategy 2.2.7. The second line in the above proof diagram asserts that we must prove that $k\beta$ is an upper bound for the set A. The third line states that we will assume that c is an upper bound for A. We will then have to prove that $k\beta \leq c$. Since every element in A has the form kx for $x \in S$, to say that c is an upper bound for A just means that $kx \leq c$ for all $x \in S$. We now have all of the necessary ingredients to compose a correct proof of the lemma.

Proof of Lemma 2.2.8. We have that $S \subseteq \mathbb{R}$ is nonempty and bounded above. The completeness axiom asserts that S has a least upper bound β, and so $\beta = \sup(S)$. Thus, $x \leq \beta$ for all $x \in S$. Let $k \geq 0$. We note that every element of $A = \{kx : x \in S\}$ has the form kx for an $x \in S$. There are two cases to consider.

CASE 1: Assume $k = 0$. Then $A = \{0\}$ and we clearly have $\sup(A) = k \sup(S)$.

CASE 2: Assume $k > 0$. We first prove that $k\beta$ is an upper bound for A. Since $x \leq \beta$ for all $x \in S$ and $k > 0$, we have that $kx \leq k\beta$ for all $x \in S$. Therefore, $k\beta$ is an upper bound for A.

Let c be an upper bound for A. We shall now prove that $k\beta \leq c$. Since c is an upper bound for A, we have that $kx \leq c$ for all $x \in S$. Because $k > 0$, we see that $x \leq \frac{c}{k}$ for all $x \in S$. Thus, $\frac{c}{k}$ is an upper bound for S. Since β is the smallest upper bound for S, we conclude that $\beta \leq \frac{c}{k}$ and so, $k\beta \leq c$. Therefore, $\sup(A) = k\beta$. □

2.2.2   Proofs on the Infimum of a Set

Remark 2.2.5(2) motivates our next strategy for proving equations of the form $\inf(S) = \alpha$.

Proof Strategy 2.2.9. Given a real number α and a nonempty $S \subseteq \mathbb{R}$, to prove that $\inf(S) = \alpha$ we may use the two-step proof diagram

$$\text{Step (1): Prove } \alpha \leq x \text{ for all } x \in S.$$
$$\text{Step (2): Assume } a \text{ is a lower bound for } S.$$
$$\text{Prove } a \leq \alpha.$$

Our next theorem shows that completeness axiom implies that if a set $S \subseteq \mathbb{R}$ is nonempty and bounded below, then there is a real number α that satisfies the equation $\alpha = \inf(S)$. Before reading the proof of next theorem, one should read Remark 2.2.5 and proof strategy 2.2.9.

Theorem 2.2.10. *Every nonempty $S \subseteq \mathbb{R}$ that is bound below, has a greatest lower bound.*

Proof. Assume that $S \subseteq \mathbb{R}$ is nonempty and has a lower bound a. Let $S^* = \{-x : x \in S\}$. Then every element of S^* has the form $-x$ for an $x \in S$. Since a is an lower bound for S, we have $a \leq x$ for all $x \in S$. Thus, $-x \leq -a$ for all $x \in S$. Therefore, $-a$ is an upper bound for S^*. By the completeness axiom, S^* has a least upper bound β. Thus, $\sup(S^*) = \beta$; that is, $-x \leq \beta$ for all $x \in S$ and β is the smallest such upper bound.

We now prove that $\inf(S) = -\beta$. First we prove that $-\beta$ is a lower bound for S. Since $-x \leq \beta$ for all $x \in S$, it follows that $-\beta \leq x$ for all $x \in S$. Thus, $-\beta$ is a lower bound for S.

To prove that $-\beta$ is the largest lower bound for S. Let a be a lower bound for S. By the argument in the first paragraph, we see that $-a$ is an upper bound for the set S^*. Since β is the least such upper bound for S^*, it follows that $\beta \leq -a$. Hence, $a \leq -\beta$. Therefore, $\inf(S) = -\beta$, that is, $-\beta$ is the greatest lower bound for S. □

Assumption Strategy 2.2.11. Let $S \subseteq \mathbb{R}$ be nonempty and let $\alpha \in \mathbb{R}$. Suppose that you are *assuming* that $\inf(S) = \alpha$. Then (1) $\alpha \leq x$ for all $x \in S$, and (2) whenever a is a lower bound for S, you can conclude that $a \leq \alpha$.

Lemma 2.2.12. Let $S \subseteq \mathbb{R}$ be nonempty and bounded below, and let $k > 0$. Define the set A by $A = \{kx : x \in S\}$. Then the set A is bounded below and $\inf(A) = k \inf(S)$.

Proof Analysis. Since the set S is bounded below there is a real number α satisfying $\alpha = \inf(S)$ by the completeness axiom. So, we must prove that $\inf(A) = k\alpha$. First note that every element in A has the form kx for $x \in S$. Thus, to prove that $k\alpha$ is an lower bound for A, we just need to prove that $k\alpha \leq kx$ for all $x \in S$. Appealing to proof strategy 2.2.9, we construct the proof diagram:

> Assume $\alpha = \inf(S)$.
>> Prove $k\alpha \leq kx$ for all $x \in S$.
>> Assume c is an lower bound for A.
>>> Prove $c \leq k\alpha$.

The first line of the above proof diagram indicates that, in our proof, we will be assuming $\alpha = \inf(S)$. Thus, we can use assumption strategy 2.2.11. The second line in the above proof diagram asserts that we must prove that $k\alpha$ is an lower bound for the set A. The third line states that we will assume that c is an lower bound for A. We will then have to prove that $c \leq k\alpha$. Since every element in A has the form kx for $x \in S$, to say that c is an lower bound for A just means that $c \leq kx$ for all $x \in S$. We now have all of the necessary ingredients to compose a correct proof of the lemma.

Proof of Lemma 2.2.12. We have that $S \subseteq \mathbb{R}$ is nonempty and bounded below. Theorem 2.2.10 implies that S has a greatest lower bound α, and so $\alpha = \inf(S)$. Thus, $\alpha \leq x$ for all $x \in S$. We are given that $k > 0$. We note that every element of $A = \{kx : x \in S\}$ has the form kx for an $x \in S$. First we prove that $k\alpha$ is a lower bound for A. Since $\alpha \leq x$ for all $x \in S$ and $k > 0$, we have that $k\alpha \leq kx$ for all $x \in S$. Therefore, $k\alpha$ is a lower bound for A.

Let c be a lower bound for A. We will now prove that $c \leq k\alpha$. Since c be a lower bound for A, we see that $c \leq kx$ for all $x \in S$. Because $k > 0$, we have that $\frac{c}{k} \leq x$ for all $x \in S$. Thus, $\frac{c}{k}$ is a lower bound for S. Since α is the largest lower bound for S, we conclude that $\frac{c}{k} \leq \alpha$. Hence, $c \leq k\alpha$ and therefore, $\inf(A) = k\alpha$. □

Theorem 2.2.13. Let $S \subseteq \mathbb{R}$ be nonempty and bounded, and let $k \in \mathbb{R}$. Define the set kS by $kS = \{kx : x \in S\}$. Then the set kS is bounded and

(a) if $k \geq 0$, then $\sup(kS) = k \sup(S)$ and $\inf(kS) = k \inf(S)$,
(b) if $k < 0$, then $\sup(kS) = k \inf(S)$ and $\inf(kS) = k \sup(S)$.

Proof. Lemmas 2.2.8 and 2.2.12 imply (a). The proof of statement (b) is an exercise. □

Theorem 2.2.14. Let $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ be non-empty sets. Suppose for all $x \in A$ and all $y \in B$ we have that $x \leq y$. Then A is bounded above, B is bounded below and $\sup(A) \leq \inf(B)$.

Proof. Suppose that

$$x \leq y \text{ for all } x \in A \text{ and all } y \in B. \tag{2.2}$$

Let $y \in B$ be arbitrary. From (2.2) we have that $x \leq y$ for all $x \in A$. Thus, y is an upper bound for A (see Figure 2.3). The completeness axiom implies that $\gamma = \sup(A)$ exists. Since y is an upper bound for A and γ is the least such upper bound, we conclude that $\gamma \leq y$. Because y is an arbitrary element in B, it follows that $\gamma \leq y$ for all $y \in B$. Thus, γ is a lower bound for B. Theorem 2.2.10 implies that $\delta = \inf(B)$ exists. Because γ is a lower bound for B and δ is the greatest such lower bound, we see that $\gamma \leq \delta$. Therefore, $\sup(A) \leq \inf(B)$. $\square$
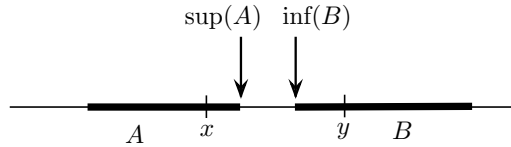

Figure 2.3: An illustration for the proof of Theorem 2.2.14

The proof of Theorem 2.2.14 easily adapts to prove our next result.

Theorem 2.2.15. Let $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ be non-empty sets. Let $c \in \mathbb{R}$. Suppose for all $x \in A$ and all $y \in B$ we have that $x - y < c$. Then A is bounded above, B is bounded below, and $\sup(A) - \inf(B) \leq c$.

Definition 2.2.16 (Maximum Element and Minimum Element)**.** Let $S \subseteq \mathbb{R}$.

- Suppose $b = \sup(S)$. If $b \in S$, then b is called the **maximum element** of S, and we write $b = \max(S)$.

- Suppose $a = \inf(S)$. If $a \in S$, then a is called the **minimum element** of S, and we write $a = \min(S)$.

For example, let $S = [2, 5]$. Because $\sup(S) = 5$ and $5 \in S$, we observe that $\max(S) = 5$. Since $\inf(S) = 2$ and $2 \in S$, we see that $\min(S) = 2$. For another example, let $T = \{\frac{1}{n} : n \in \mathbb{N}\}$. Because $\sup(T) = 1$ and $1 \in T$, we have that $\max(T) = 1$. Since $\inf(T) = 0$ and $0 \notin T$, we conclude that $\min(T)$ is undefined.

Definition 2.2.17. A function $f \colon D \to \mathbb{R}$ is **bounded** if the set $f[D]$ is bounded.

Remark 2.2.18. Suppose $f \colon D \to \mathbb{R}$ is bounded. Thus, the set $f[D] = \{f(x) : x \in D\}$ is bounded. Let $\beta = \sup(f[D])$ and $\alpha = \inf(f[D])$. Then $\alpha \leq f(x) \leq \beta$ for all $x \in D$.

Remark 2.2.19. Let $f \colon D \to \mathbb{R}$. Theorem 2.2.2 implies that the are two equivalent ways of saying that the set $f[D] = \{f(x) : x \in D\}$ is bounded, namely:

(1) there are real numbers a, b such that $a \leq f(x) \leq b$ for all $x \in D$,

(2) there is a real number $M > 0$ such that $|f(x)| \leq M$ for all $x \in D$.

Theorem 2.2.20. If $f \colon D \to \mathbb{R}$ and $g \colon D \to \mathbb{R}$ are bounded, then $(f + g) \colon D \to \mathbb{R}$ is bounded and

(a) $\sup((f + g)[D]) \leq \sup(f[D]) + \sup(g[D])$,

(b) $\inf(f[D]) + \inf(g[D]) \leq \inf((f + g)[D])$.

Proof. Assume $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ are bounded. We shall prove only (b) and leave (a) as an exercise. Since $f[D]$ and $g[D]$ are bounded below, let $\varepsilon = \inf(f[D])$ and $\delta = \inf(g[D])$. Thus, (*) $\varepsilon \le f(x)$ for all $x \in D$, and (**) $\delta \le g(x)$ for all $x \in D$. Inequalities (*) and (**) imply that $\varepsilon + \delta \le f(x) + g(x)$ for all $x \in D$. Since $(f + g)(x) = f(x) + g(x)$, we conclude that $\varepsilon + \delta \le (f + g)(x)$ for all $x \in D$. Therefore, $\varepsilon + \delta$ is a lower bound for $(f + g)[D]$. Thus, $\gamma = \inf((f + g)[D])$ exists. Since γ is the greatest lower bound for $(f + g)[D]$, we see that $\varepsilon + \delta \le \gamma$ and therefore, $\inf(f[D]) + \inf(g[D]) \le \inf((f + g)[D])$. $\qquad\qquad\square$

Theorem 2.2.21. Let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be functions. Assume that $f(x) \le g(y)$ for all $x, y \in D$. Then $\sup(f[D]) \le \inf(g[D])$.

Proof. Assume that $f(x) \le g(y)$ for all $x, y \in D$. We will show that $a \le b$ for all $a \in f[D]$ and all $b \in g[D]$. Let $a \in f[D]$ and $b \in g[D]$. Since $a \in f[D]$, there is an $x \in D$ such that $a = f(x)$. Also, since $b \in g[D]$, there is a $y \in D$ such that $b = g(y)$. By our assumption, we see that $a = f(x) \le g(y) = b$. Thus, $a \le b$ for all $a \in f[D]$ and all $b \in g[D]$. Theorem 2.2.14 now implies that $\sup(f[D]) \le \inf(g[D])$. $\qquad\qquad\square$
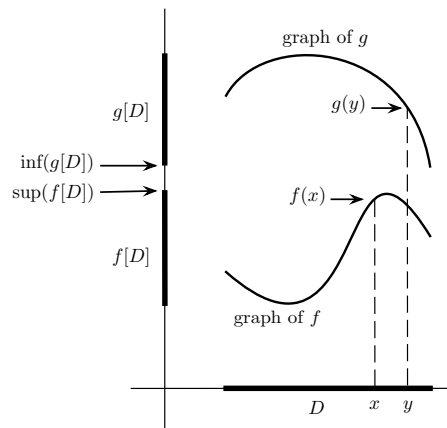

Figure 2.4: Representation of Theorem 2.2.21

2.2.3 Alternative Proof Strategies

The proofs of our earlier results on the supremum and infimimum relied on Remark 2.2.5 in which we observed that the equation $\beta = \sup(S)$ means that (a) β is an upper bound for S and (b) β is the smallest upper bound for S. Given (a), another way to say (b) is that "every number $r < \beta$ is not an upper bound for S." A similar observation can be made concerning the infimum of a set.

Remark 2.2.22. Let $S \subseteq \mathbb{R}$ be nonempty and let $\alpha, \beta \in \mathbb{R}$. Then

1. $\beta = \sup(S)$ if and only if the following two conditions hold:

    (a) For all $x \in S$, $x \le \beta$.
    (b) For all real numbers r if $r < \beta$, then there is an $x \in S$ so that $r < x$.

2. $\alpha = \inf(S)$ if and only if the following two conditions hold:

    (a) For all $x \in S$, $\alpha \le x$.
    (b) For all real numbers q if $\alpha < q$, then there is an $x \in S$ so that $x < q$.

Remark 2.2.22(1) motivates the following alternative strategies for dealing with an equation of the form $\sup(S) = \beta$.
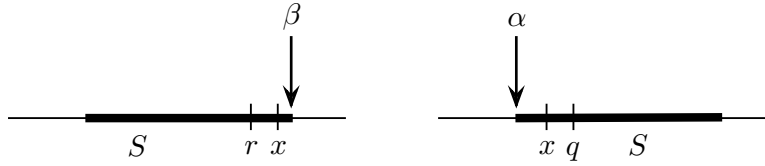
Figure 2.5: Illustrations for parts 1 and 2 of Remark 2.2.22, respectively

Proof Strategy 2.2.23. Given a real number β and a nonempty $S \subseteq \mathbb{R}$, to prove that $\sup(S) = \beta$ we may use the two-step proof diagram

> Step (1): Prove $x \leq \beta$ for all $x \in S$.
> Step (2): Assume $r < \beta$.
>   Prove $r < x$ for some $x \in S$.

Assumption Strategy 2.2.24. Let $S \subseteq \mathbb{R}$ be nonempty and let $\beta \in \mathbb{R}$. Suppose that you are *assuming* that $\sup(S) = \beta$. Then (1) $x \leq \beta$ for all $x \in S$, and (2) whenever $r < \beta$ there is an $x \in S$ such that $r < x$.

We will present two proofs of our next theorem. In the first proof we shall apply strategies 2.2.23 and 2.2.24. In our second proof we apply strategies 2.2.6 and 2.2.7.

Theorem 2.2.25. Let A and B be nonempty subsets of $\mathbb{R}$ that are both bounded above. Let $C = \{x + y : x \in A \text{ and } y \in B\}$. Then C is bounded above and $\sup(C) = \sup(A) + \sup(B)$.

Proof Analysis. Since the sets A and B are bounded above there are a real numbers α and β satisfying $\alpha = \sup(A)$ and $\beta = \sup(B)$ by the completeness axiom. So, we must prove that $\sup(C) = \alpha + \beta$. First note that every element in C has the form $x + y$ for $x \in A$ and $y \in B$. Thus, to prove that $\alpha + \beta$ is an upper bound for C, we just need to prove that $x + y \leq \alpha + \beta$ for all $x \in A$ and $y \in B$. Appealing to proof strategy 2.2.23, we construct the following proof diagram:

> Assume $\alpha = \sup(A)$ and $\beta = \sup(B)$.
>   Prove $x + y \leq \alpha + \beta$ for all $x \in A$ and $y \in B$.
>   Assume $s < \alpha + \beta$.
>   Prove $s < x + y$ for an $x \in A$ and a $y \in B$.

The first line of the above proof diagram indicates that we will be assuming $\alpha = \sup(A)$ and $\beta = \sup(B)$. Thus, we can use assumption strategy 2.2.24. The second line in the above proof diagram asserts that we must prove that $\alpha + \beta$ is an upper bound for the set C. The third line states that we will assume that s is a value satisfying $s < \alpha + \beta$. We will then have to prove that that there is an element in C that is larger than s. Since every element in C has the form $x + y$ for $x \in A$ and $y \in B$, we just need to find an $x \in A$ and a $y \in B$ such that $s < x + y$. We now have all of the necessary ingredients to compose a correct proof of the theorem.

First Proof of Theorem 2.2.25. Let $\alpha = \sup(A)$ and $\beta = \sup(B)$. We shall prove that $\sup(C) = \alpha + \beta$. To do this, we shall first show that $\alpha + \beta$ is an upper bound for C. Let $z \in C$. So $z = x + y$ for some $x \in A$ and some $y \in B$. Since $x \leq \alpha$ and $y \leq \beta$, we have that $z = (x + y) \leq (\alpha + \beta)$. So $\alpha + \beta$ is an upper bound for C.

To prove that $\alpha + \beta$ is the least upper bound for C, assume that $s < \alpha + \beta$. Thus, $s - \beta < \alpha$. Since $\alpha = \sup(A)$, there is an $x \in A$ such that $s - \beta < x$. Thus, $s - x < \beta$. Since $\beta = \sup(B)$, there is an $y \in B$ such that $s - x < y$. Hence $s < x + y$ and $x + y \in C$. Therefore, $\sup(C) = \alpha + \beta$. $\square$

We will now give another proof of Theorem 2.2.25 using strategies 2.2.6 and 2.2.7. Using these strategies, the logical structure of our second proof will be as follows:

Assume $\alpha = \sup(A)$ and $\beta = \sup(B)$.
Prove $x + y \le \alpha + \beta$ for all $x \in A$ and $y \in B$.
Assume c is an upper bound for C.
Prove $\alpha + \beta \le c$.

Second Proof of Theorem 2.2.25. Let $\alpha = \sup(A)$ and $\beta = \sup(B)$. We shall prove that $\sup(C) = \alpha + \beta$. To do this, we shall first show that $\alpha + \beta$ is an upper bound for C. Let $z \in C$. So $z = x + y$ for some $x \in A$ and some $y \in B$. Since $x \le \alpha$ and $y \le \beta$, we have that $z = (x + y) \le (\alpha + \beta)$. So $\alpha + \beta$ is an upper bound for C.

To prove that $\alpha + \beta$ is the least upper bound for C, suppose that c is an upper bound for C. We shall prove that $\alpha + \beta \le c$. Since c is an upper bound for C, we have that

$$x + y \le c \text{ for all } x \in A \text{ and all } y \in B. \tag{2.3}$$

Let $y \in B$ be arbitrary. From (2.3) we have that $x + y \le c$ for all $x \in A$. Hence, $x \le c - y$ for all $x \in A$. Thus, $c - y$ is an upper bound for A. Since $\alpha = \sup(A)$, we infer that $\alpha \le c - y$. Since y was arbitrary, it follows that $y \le c - \alpha$ for all $y \in B$. Therefore, $c - \alpha$ is an upper bound for B. Since $\beta = \sup(B)$, we conclude that $\beta \le c - \alpha$. Thus, $\alpha + \beta \le c$. Therefore, $\sup(C) = \alpha + \beta$. $\quad\square$

Remark 2.2.22(2) also provides us with alternative strategies for working with an equation of the form $\inf(S) = \alpha$.

Proof Strategy 2.2.26. Given a real number α and a nonempty $S \subseteq \mathbb{R}$, to prove that $\inf(S) = \alpha$ we may use the two-step proof diagram

Step (1): Prove $\alpha \le x$ for all $x \in S$.
Step (2): Assume $\alpha < q$.
Prove $x < q$ for some $x \in S$.

Assumption Strategy 2.2.27. Let $S \subseteq \mathbb{R}$ be nonempty and let $\alpha \in \mathbb{R}$. Suppose that you are *assuming* that $\inf(S) = \alpha$. Then (1) $\alpha \le x$ for all $x \in S$, and (2) whenever $\alpha < q$ there is an $x \in S$ such that $x < q$.

Exercises 2.2 ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

1. For each of the follows subsets S, answer the questions: Is the set S bounded above? Is the set S bounded below?

(a) $S = [2, 5]$
(b) $S = [2, 5)$
(c) $S = (2, \infty)$
(d) $S = \mathbb{N}$
(e) $S = \{x \in \mathbb{R} : (x^2 + 1)^{-1} > \frac{1}{2}\}$
(f) $S = \{\frac{1}{n} : n \in \mathbb{N}\}$
(g) $S = \{q \in \mathbb{Q} : 0 \le q \le \sqrt{2}\}$
(h) $S = \{x \in \mathbb{R} : |2x + 1| < 5\}$.

2. For each of the follows subsets S of $\mathbb{R}$ identify the $\sup(S)$ and $\inf(S)$, if they exist.

(a) $S = [2, 5]$
(b) $S = [2, 5)$
(c) $S = (2, \infty)$
(d) $S = \mathbb{N}$
(e) $S = \{x \in \mathbb{R} : (x^2 + 1)^{-1} > \frac{1}{2}\}$
(f) $S = \{\frac{1}{n} : n \in \mathbb{N}\}$
(g) $S = \{q \in \mathbb{Q} : 0 \le q \le \sqrt{2}\}$
(h) $S = \{x \in \mathbb{R} : |2x + 1| < 5\}$.

3. For each of the follows subsets S identify the $\max(S)$ and $\min(S)$ if they exist.

   (a) $S = \{2, 5, 6\}$ (b) $S = \mathbb{N}$
   (c) $S = (2, \infty)$ (d) $S = \{q \in \mathbb{Q} : 0 \leq q \leq \sqrt{2}\}$.

4. Suppose $S \subseteq \mathbb{R}$ is nonempty and bounded. Let $A \subseteq S$ be nonempty. Prove that A is bounded. Then prove that $\sup(A) \leq \sup(S)$ and $\inf(S) \leq \inf(A)$.

5. Let $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ be non-empty bounded sets. Suppose that $\sup(A) = \inf(B) = \ell$, where ℓ is this common value. Prove that $x \leq \ell \leq y$ for all $x \in A$ and all $y \in B$.

6. Suppose $S \subseteq \mathbb{R}$ is nonempty and bounded. Let $\beta = \sup(S)$. Prove that for all $\varepsilon > 0$ there exists an $x \in S$ such that $\beta - \varepsilon < x$.

7. Suppose $S \subseteq \mathbb{R}$ is nonempty and bounded. Let $\alpha = \inf(S)$. Prove that for all $\varepsilon > 0$ there exists an $x \in S$ such that $x < \alpha + \varepsilon$.

8. Complete the proof of Lemma 2.2.13, that is, suppose that $S \subseteq \mathbb{R}$ is bounded and nonempty let $k < 0$. Prove that

   (a) $\sup(kS) = k \inf(S)$
   (b) $\inf(kS) = k \sup(S)$.

9. Suppose $S \subseteq \mathbb{R}$ is nonempty and bounded. Let $k \in \mathbb{R}$. Define $A = \{k + x : x \in S\}$. Prove that

   (a) $\sup(A) = k + \sup(S)$
   (b) $\inf(A) = k + \inf(S)$.

10. Let S be a bounded (nonempty) subset of $\mathbb{R}$. Let $k > 0$ and let $c \in \mathbb{R}$. Define $A = \{kx + c : x \in S\}$. Prove that A is bounded, and then prove the following

    (a) $\sup(A) = k \sup(S) + c$,
    (b) $\inf(A) = k \inf(S) + c$.

11. Suppose $S \subseteq \mathbb{R}$ and $T \subseteq \mathbb{R}$ are nonempty and bounded. Prove that $S \cup T$ is bounded, and then prove the following

    (a) $\sup(S \cup T) = \max\{\sup(S), \sup(T)\}$.
    (b) $\inf(S \cup T) = \min\{\inf(S), \inf(T)\}$

12. Prove Theorem 2.2.15.

13. Prove part (a) of Theorem 2.2.20.

14. Let $S \subseteq \mathbb{R}^+$ and $T \subseteq \mathbb{R}^+$ be nonempty and bounded above. Let $P = \{xy : x \in S \text{ and } y \in T\}$. Prove that $\sup(P) = \sup(S) \cdot \sup(T)$.

15. Let $f : D \to \mathbb{R}$ be bounded. Let $E \subseteq D$ be nonempty. Prove that the set $f[E]$ is bounded. Then prove that $\sup(f[E]) \leq \sup(f[D])$ and $\inf(f[D]) \leq \inf(f[E])$.

16. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be bounded. Assume that $f(x) \leq g(x)$ for all $x \in D$. Prove that $\sup(f[D]) \leq \sup(g[D])$ and $\inf(f[D]) \leq \inf(g[D])$.

17. Let A and B be nonempty subsets of $\mathbb{R}$ which are both bounded below. Define the set $C = \{x + y : x \in A \text{ and } y \in B\}$. Prove that C is bounded below and that $\inf(C) = \inf(A) + \inf(B)$.

18. Let $S \subseteq \mathbb{R}^+$ and $T \subseteq \mathbb{R}^+$ be nonempty and bounded below. Let $P = \{xy : x \in S \text{ and } y \in T\}$. Prove that $\inf(P) = \inf(S) \cdot \inf(T)$.

19. Let $I \subseteq \mathbb{R}$ be nonempty. Prove that I is an interval if and only if for all x and y in I, if $x < y$, then $[x, y] \subseteq I$.

Exercise Notes: For Exercise 1, $|2x + 1| < 5$ is equivalent to $-5 < 2x + 1 < 5$. For Exercises 6 and 7, see Remark 2.2.22. For Exercises 9 and 10 review the analysis and proofs for Lemmas 2.2.8 and 2.2.12. For Exercise 14 review the analysis and proof of Theorem 2.2.25. For Exercise 17, review strategies 2.2.26 and 2.2.27. For Exercise 18, first show that $\alpha = \inf(S) \geq 0$ and $\beta = \inf(T) \geq 0$; and review strategies 2.2.9 and 2.2.11. Let c be a lower bound for P. If $c \leq 0$, then clearly $c \leq \alpha\beta$. Suppose $c > 0$ and let $y \in T$. Prove that $\frac{c}{y}$ is a lower bound for S.

2.3 The Archimedean Property

The completeness axiom implies the *Archimedean property*, which asserts that each real number is strictly less than some natural number.

Theorem 2.3.1 (Archimedean Property for $\mathbb{R}$). For each $x \in \mathbb{R}$ there is an $n \in \mathbb{N}$ such that $x < n$.

Proof. Let $x \in \mathbb{R}$. We prove that there exists a natural number $n \in \mathbb{N}$ such that $x < n$. Suppose, for a contradiction, that $n \leq x$ for all $n \in \mathbb{N}$. Thus, $\mathbb{N} \subseteq \mathbb{R}$ is bounded above. Hence, by the completeness axiom, $\beta = \sup(\mathbb{N})$ exists. Now because $\beta - 1 < \beta$ there is an $m \in \mathbb{N}$ such that $\beta - 1 < m$ (see Remark 2.2.22(1b)). Therefore, $\beta < m + 1 = n \in \mathbb{N}$; contradicting the fact that β is an upper bound for $\mathbb{N}$. This contradiction completes the proof. $\square$

In our next theorem, we show that the Archimedean property implies two useful results.

Theorem 2.3.2. Each of the following statements hold:

(a) For all $x \in \mathbb{R}$ and $y \in \mathbb{R}$, if $x > 0$, then there is an $n \in \mathbb{N}$ such that $y < nx$.

(b) For all $x \in \mathbb{R}$, if $x > 0$, then there is an $n \in \mathbb{N}$ such that $0 < 1/n < x$.

Proof. We first prove (a). Let $x, y \in \mathbb{R}$ where $x > 0$. Consider the real number y/x. By Theorem 2.3.1, there is an $n \in \mathbb{N}$ such that $y/x < n$. We conclude that $y < nx$ and the proof of (a) is complete. Now to prove (b), let $x > 0$. From (a) (where we take $y = 1$), we conclude that there is an $n \in \mathbb{N}$ such that $0 < 1 < nx$. Thus, $0 < 1/n < x$. $\square$

Lemma 2.3.3. For all $x \in \mathbb{R}$ there exists an integer m such that $m - 1 \leq x < m$.

Proof. Let $x \in \mathbb{R}$. We will show that there exists an $m \in \mathbb{Z}$ such that $m - 1 \leq x < m$. If $x \in \mathbb{Z}$, then $m = x + 1 \in \mathbb{Z}$ and $m - 1 \leq x < m$. Now suppose that $x \notin \mathbb{Z}$ and so $x \neq 0$. First suppose that $x > 0$. If $x < 1$, then $m = 1$ is as required. If $x \geq 1$, then (by Theorem 2.3.1 and the Well-Ordering Principle) let m be the least natural number such that $x < m$. Since the natural number $m - 1$ is less that m, it follows that $m - 1 \leq x$. Therefore, $m - 1 \leq x < m$.

Now suppose that $x < 0$, then $-x > 0$ and the argument in the previous paragraph implies that there is an $n \in \mathbb{N}$ such that $n - 1 \leq -x < n$. Thus, by applying properties of inequalities, we conclude that $-n < x \leq 1 - n$. Since $x \notin \mathbb{Z}$ we see that $-n < x < 1 - n$. Therefore, for $m = 1 - n$ we have that $m \in \mathbb{Z}$, $m - 1 = -n$ and $m - 1 \leq x < m$. $\square$

Lemma 2.3.4. For all $a, b \in \mathbb{R}$ if $1 < b - a$, then there exists an $m \in \mathbb{Z}$ such that $a < m < b$.

Proof. Let $a, b \in \mathbb{R}$. Assume that $1 < b - a$. We will show that there exists an $m \in \mathbb{Z}$ such that $a < m < b$. Since $1 < b - a$, we have that (i) $1 + a < b$. By Lemma 2.3.3, there is an $m \in \mathbb{Z}$ such that (ii) $m - 1 \leq a$ and (iii) $a < m$. Note that (i) and (ii) imply that $m < b$. Thus, (iii) implies that $a < m < b$. $\square$