2.4 Multiplicative Functions

Definition 2.4.1. We write $f\colon A \to B$ to mean that f is a **function** from the set A to the set B, that is, for every element $x \in A$ there is exactly one element $f(x)$ in B. The set A is the **domain** of the function f and the set B is the **co-domain** of f.

Let $n = 24$. The positive divisors of n are $1, 2, 3, 4, 6, 8, 12, 24$. So the number of such divisors is 8. The sum of all these divisors is $1 + 2 + 3 + 4 + 6 + 8 + 12 + 24 = 60$. In this section we will be investigating the number of positive divisors of any given positive integer n, and the sum of all these divisors.

Definition 2.4.2. Let n be a positive integer. Then $d(n)$ equals the number of positive divisors of n (including 1 and n). Also, $\sigma(n)$ equals the sum of all the positive divisors of n (including 1 and n).

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$d(n)$	1	2	2	3	2	4	2	4	3	4	2	6	2	4
$\sigma(n)$	1	3	4	7	6	12	8	15	13	18	12	28	14	24

At the top of the page we evaluated $d(24) = 8$ and $\sigma(24) = 60$. Because $24 = 3 \cdot 8$, we note that $d(3) = 2$ and $d(8) = 4$. Thus, we observe that $d(3 \cdot 8) = d(3) \cdot d(8)$. In addition, we have that $\sigma(3) = 4$ and $\sigma(8) = 15$. We also observe $\sigma(3 \cdot 8) = \sigma(3) \cdot \sigma(8)$. We shall prove that for all natural numbers m and n, if $(m, n) = 1$ then

$$d(m \cdot n) = d(m) \cdot d(n) \tag{2.9}$$

$$\sigma(m \cdot n) = \sigma(m) \cdot \sigma(n) \tag{2.10}$$

On the other hand, we have that $d(8) = 4$ and $\sigma(8) = 15$. Yet $8 = 2 \cdot 4$, but $d(2 \cdot 4) \neq d(2) \cdot d(4)$ and $\sigma(2 \cdot 4) \neq \sigma(2) \cdot \sigma(4)$. Thus, (2.9) and (2.10) hold only if m and n are relatively prime.

Definition 2.4.3. A function $f\colon \mathbb{N} \to \mathbb{N}$ is said to be **multiplicative** if for all pairs of *relatively prime* natural numbers m and n, we have that $f(m \cdot n) = f(m) \cdot f(n)$.

In other words, $f\colon \mathbb{N} \to \mathbb{N}$ is multiplicative if $(m, n) = 1$ then $f(m \cdot n) = f(m) \cdot f(n)$ for all natural numbers m and n.

Example 1. Suppose $f\colon \mathbb{N} \to \mathbb{N}$ is a multiplicative function. Then $f(126) = f(2 \cdot 3^2 \cdot 7) = f(2) \cdot f(3^2) \cdot f(7)$.

Definition 2.4.4. A function $f\colon \mathbb{N} \to \mathbb{N}$ is said to be **completely multiplicative** if for *all* pairs of natural numbers m and n, we have that $f(m \cdot n) = f(m) \cdot f(n)$.

Example 2.

1. The function $f\colon \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 1$ is completely multiplicative because $f(m \cdot n) = 1 = 1 \cdot 1 = f(m) \cdot f(n)$.
2. The function $h\colon \mathbb{N} \to \mathbb{N}$ defined by $h(n) = n$ is completely multiplicative because $h(m \cdot n) = m \cdot n = h(m) \cdot h(n)$.

Remark. If a function is completely multiplicative, then it is also multiplicative.

Summation Notation

Given a function $f\colon \mathbb{N} \to \mathbb{N}$ we write $\sum_{n=1}^{k} f(n)$ as shorthand for the sum $f(1)+f(2)+\cdots+f(k)$; that is,

$$\sum_{n=1}^{k} f(n) = f(1) + f(2) + \cdots + f(k).$$

More generally, given natural numbers $m \leq k$ we shall write $\sum_{n=m}^{k} f(n)$ as shorthand for the sum $f(m) + f(m+1) + \cdots + f(k)$; that is,

$$\sum_{n=m}^{k} f(n) = f(m) + f(m+1) + \cdots + f(k).$$

Another way to denote the above sum is with the notation $\sum_{m \leq n \leq k} f(n)$. However, we will also be interested in adding just the $f(n)$'s for $m \leq n \leq k$ when n has a special property. For example, to add the $f(n)$'s for just the even n where $m \leq n \leq k$, we use the notation $\sum_{\substack{m \leq n \leq k \\ n \text{ is even}}} f(n)$. For example,

$$\sum_{\substack{1 \leq n \leq 8 \\ n \text{ even}}} f(n) = f(2) + f(4) + f(6) + f(8).$$

For another example, suppose we want to add the $f(n)$'s for just the natural numbers n that evenly divide 16. We shall use the notation

$$\sum_{n \mid 16} f(n) = f(1) + f(2) + f(4) + f(8) + f(16).$$

Here are some other examples:

$$\sum_{\substack{d_1 \mid 6 \\ d_2 \mid 4}} f(d_1 \cdot d_2) = f(1 \cdot 1) + f(2 \cdot 1) + f(3 \cdot 1) + f(6 \cdot 1) + f(1 \cdot 2) + f(2 \cdot 2)$$
$$+ f(3 \cdot 2) + f(6 \cdot 2) + f(1 \cdot 4) + f(2 \cdot 4) + f(3 \cdot 4) + f(6 \cdot 4)$$

$$\sum_{\substack{d_1 \mid 6 \\ d_2 \mid 6 \\ d_1 d_2 = 6}} f(d_1) f(d_2) = f(1)f(6) + f(2)f(3) + f(3)f(2) + f(6)f(1).$$

Example 3. Notice that

$$\begin{aligned}\left(\sum_{d_1\,|\,6} f(d_1)\right)\left(\sum_{d_2\,|\,4} f(d_2)\right) &= [\,f(1)+f(2)+f(3)+f(6)\,][\,f(1)+f(2)+f(4)\,]\\ &= f(1)f(1)+f(2)f(1)+f(3)f(1)+f(6)f(1)\\ &\quad + f(1)f(2)+f(2)f(2)+f(3)f(2)+f(6)f(2)\\ &\quad + f(1)f(4)+f(2)f(4)+f(3)f(4)+f(6)f(4)\\ &= \sum_{\substack{d_1\,|\,6\\ d_2\,|\,4}} f(d_1)f(d_2).\end{aligned}$$

Therefore, by algebra, we have that $\left(\sum_{d_1\,|\,6} f(d_1)\right)\left(\sum_{d_2\,|\,4} f(d_2)\right) = \sum_{\substack{d_1\,|\,6\\ d_2\,|\,4}} f(d_1)f(d_2)$.

Example 4. Since $(3,4)=1$ and $12 = 3\cdot 4$, Theorem 2.3.4 implies that

> for any $d>0$, we have that $d\,|\,12$ if and only if there are unique positive integers d_1 and d_2 such that $d = d_1d_2$, $d_1\,|\,3$, and $d_2\,|\,4$.

Thus, we obtain the following identity:

$$\begin{aligned}\sum_{\substack{d_1\,|\,3\\ d_2\,|\,4}} f(d_1\cdot d_2) &= f(1\cdot 1)+f(1\cdot 2)+f(1\cdot 4)+f(3\cdot 1)+f(3\cdot 2)+f(3\cdot 4)\\ &= f(1)+f(2)+f(4)+f(3)+f(6)+f(12)\\ &= \sum_{d\,|\,12} f(d).\end{aligned}$$

Therefore, it follows from Theorem 2.3.4 that $\sum_{\substack{d_1\,|\,3\\ d_2\,|\,4}} f(d_1\cdot d_2) = \sum_{d\,|\,12} f(d)$.

More generally, we have the following theorem.

Theorem 2.4.5. Let $f\colon \mathbb{N}\to\mathbb{N}$ be any function. Suppose that m and n are natural numbers such that $(m,n)=1$. Then $\sum_{\substack{d_1\,|\,m\\ d_2\,|\,n}} f(d_1\cdot d_2) = \sum_{d\,|\,(mn)} f(d)$.

Proof. Consider the following list of all products of natural numbers d_1 and d_2 where $d_1\,|\,m$ and $d_2\,|\,n$

$$1\cdot 1,\ldots,1\cdot n,\ldots,d_1\cdot d_2,\ldots,m\cdot 1,\ldots,m\cdot n \tag{2.11}$$

and consider the following list of all natural numbers d where $d\,|\,(mn)$

$$1,\ldots,n,\ldots,d,\ldots,m,\ldots,mn \tag{2.12}$$

We will now show that the numbers in list (2.11) are distinct and are also on list (2.12). Let $d_1 \cdot d_2$ be on list (2.11). Since $d_1 \mid m$ and $d_2 \mid n$, it follows (directly from the definition of divisibility) that $(d_1 d_2) \mid (mn)$ and therefore, $d = d_1 d_2$ is on the list (2.12). Theorem 2.3.4 now implies that for any $d > 0$ we have that

> ($\star$) $d \mid (mn)$ if and only if there are unique positive integers d_1 and d_2 such that $d = d_1 d_2$, $d_1 \mid m$, and $d_2 \mid n$.

It now follows that the numbers in list (2.11) are all distinct. Furthermore, ($\star$) also implies that every number in list (2.12) is also on list (2.11). Therefore, the numbers in list (2.11) are exactly the same as the numbers in list (2.12) (except for order).

Thus, the list of values obtained by applying f to the entries in list (2.11)

$$f(1 \cdot 1), \ldots, f(1 \cdot n), \ldots, f(d_1 \cdot d_2), \ldots, f(m \cdot 1), \ldots, f(m \cdot n)$$

is exactly the same as the list of values obtained by applying f to the entries in list (2.12)

$$f(1), \ldots, f(n), \ldots, f(d), \ldots, f(m), \ldots, f(mn)$$

(except, possibly, for order). Therefore, $\displaystyle\sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f(d_1 \cdot d_2) = \sum_{d \mid (mn)} f(d).$ □

Example 5. Suppose that $f\colon \mathbb{N} \to \mathbb{N}$ is a multiplicative function. Define $g\colon \mathbb{N} \to \mathbb{N}$ by

$$g(n) = \sum_{d \mid n} f(d).$$

We will now show that $g(3)g(4) = g(12)$. First note that

$$\begin{aligned} g(3) &= \sum_{d \mid 3} f(d) = f(1) + f(3) \\ g(4) &= \sum_{d \mid 4} f(d) = f(1) + f(2) + f(4) \end{aligned}$$

Thus,

$$\begin{aligned} g(3)g(4) &= \left(\sum_{d \mid 3} f(d)\right)\left(\sum_{d \mid 4} f(d)\right) \\ &= [\, f(1) + f(3)\,]\,[\, f(1) + f(2) + f(4)\,] \\ &= f(1)f(1) + f(1)f(2) + f(1)f(4) + f(3)f(1) + f(3)f(2) + f(3)f(4) \\ &= f(1 \cdot 1) + f(1 \cdot 2) + f(1 \cdot 4) + f(3 \cdot 1) + f(3 \cdot 2) + f(3 \cdot 4) \\ &= f(1) + f(2) + f(4) + f(3) + f(6) + f(12) \\ &= \sum_{d \mid 12} f(d) \\ &= g(12) \end{aligned}$$

Since $(3,4)=1$, for any $d_1 \mid 3$ and $d_2 \mid 4$ we have that $(d_1,d_2)=1$. Using only sigma notation, we can now show that $g(3)g(4)=g(12)$ as follows:

$$\begin{aligned}
g(3)g(4) &= \left(\sum_{d_1 \mid 3} f(d_1)\right)\left(\sum_{d_2 \mid 4} f(d_2)\right) && \text{by definition of } g\\
&= \sum_{\substack{d_1 \mid 3\\ d_2 \mid 4}} f(d_1)f(d_2) && \text{by algebra.}\\
&= \sum_{\substack{d_1 \mid 3\\ d_2 \mid 4}} f(d_1d_2) && \text{because } (d_1,d_2)=1 \text{ and } f \text{ is multiplicative}\\
&= \sum_{d \mid 12} f(d) && \text{by Example 4.}\\
&= g(12) && \text{by definition of } g.
\end{aligned}$$

This completes the example.

Theorem 2.4.6 (2.15 of text)**.** Suppose that $f\colon \mathbb{N} \to \mathbb{N}$ is a multiplicative function. Define $g\colon \mathbb{N} \to \mathbb{N}$ by $g(n) = \sum_{d \mid n} f(d)$. Then g is also a multiplicative function.

Proof. Suppose that $m > 0$ and $n > 0$ are such that $(m,n)=1$. We now prove that $g(m)g(n)=g(mn)$ as follows:

$$\begin{aligned}
g(m)g(n) &= \left(\sum_{d_1 \mid m} f(d_1)\right)\left(\sum_{d_2 \mid n} f(d_2)\right) && \text{by definition of } g\\
&= \sum_{\substack{d_1 \mid m\\ d_2 \mid n}} f(d_1)f(d_2) && \text{by algebra}\\
&= \sum_{\substack{d_1 \mid m\\ d_2 \mid n}} f(d_1d_2) && \text{because } (d_1,d_2)=1 \text{ and } f \text{ is multiplicative}\\
&= \sum_{d \mid (mn)} f(d) && \text{by Theorem 2.4.5.}\\
&= g(mn) && \text{by definition of } g.
\end{aligned}$$

This completes the proof. □

Recall that $d(n)$ equals the number of positive divisors of n (including 1 and n). Also, $\sigma(n)$ equals the sum of all the positive divisors of n (including 1 and n). Recall from Example 2 the functions $f\colon \mathbb{N} \to \mathbb{N}$ defined by $f(n)=1$ and $h\colon \mathbb{N} \to \mathbb{N}$ defined by $h(n)=n$. Note that $d(6)=4$ and that $\sum_{d \mid 6} f(d) = f(1)+f(2)+f(3)+f(6) = 1+1+1+1 = 4$. Furthermore,

$\sigma(6) = 12$ and $\sum_{d\,|\,6} h(d) = h(1) + h(2) + h(3) + h(6) = 1 + 2 + 3 + 6 = 12$. In general, we see that

$$d(n) = \sum_{d\,|\,n} f(d) \tag{2.13}$$

$$\sigma(n) = \sum_{d\,|\,n} h(d) \tag{2.14}$$

because in (2.13) the right hand side adds 1 for each positive divisor d of n, and in (2.14) the right hand side adds d for each positive divisor d of n. We verified that the functions f and h are multiplicative functions in Example 2. Thus, equations (2.13), (2.14) and Theorem 2.4.6 imply the following theorem.

Theorem 2.4.7 (2.16 of text)**.** The functions d and σ are multiplicative.

Because d and σ are multiplicative functions, we can now establish our final theorem of this section.

Theorem 2.4.8 (2.17 of text)**.** Suppose that the positive integer n has the prime factorization into distinct primes

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}.$$

Then

$$d(n) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1) \tag{2.15}$$

$$\sigma(n) = \frac{p_1^{a_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{a_2+1} - 1}{p_2 - 1} \cdots \frac{p_k^{a_k+1} - 1}{p_k - 1} \tag{2.16}$$

Proof. Note that if p is a prime and $a \geq 1$, then the divisors of p^a are $1, p, p^2, \ldots, p^a$. Thus, $d(p^a) = a + 1$. Furthermore, the sum of the divisors of p^a is $1 + p + p^2 + \cdots + p^a$. Hence, $\sigma(p^a) = 1 + p + p^2 + \cdots + p^a = \frac{p^{a+1}-1}{p-1}$, where the last equality follows from the formula for a geometric sum (see any discrete mathematics text). Since d and σ are multiplicative functions, we obtain

$$d(n) = d(p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}) = d(p_1^{a_1}) d(p_2^{a_2}) \cdots d(p_k^{a_k}) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$$

$$\sigma(n) = \sigma(p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}) = \sigma(p_1^{a_1}) \sigma(p_2^{a_2}) \cdots \sigma(p_k^{a_k}) = \frac{p_1^{a_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{a_2+1} - 1}{p_2 - 1} \cdots \frac{p_k^{a_k+1} - 1}{p_k - 1}.$$

□

Example 6. Evaluate $d(500)$ and $\sigma(500)$. Since $500 = 2^2 \cdot 5^3$, we have that $d(500) = (2+1)(3+1) = 12$ and $\sigma(500) = \dfrac{2^{2+1} - 1}{2 - 1} \cdot \dfrac{5^{3+1} - 1}{5 - 1} = \dfrac{7}{1} \cdot \dfrac{624}{4} = 7 \cdot 156 = 1092$.

Definition 2.4.9. A natural number n is called a **perfect number** if $\sigma(n) - n = n$.

Exercises 2.4

Do problems #1–7, on page 43 of text.