MAT 300 - Final Exam Review Problems on Chapters 5 and 6
Final Exam on Monday Dec. 9, 9:40 to 11:00. Bacon 214A

1. Prove the following theorems:

   (a) **Theorem.** $(A \setminus B) \cap (C \setminus B) = (A \cap C) \setminus B$.
   (b) **Theorem.** $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$.
   (c) **Theorem.** If $A \setminus B \subseteq C$, then $A \setminus C \subseteq B$.

2. Define $f\colon \mathbb{Z} \to \mathbb{Z}$ by $f(n) = 3n$.

   (a) Is f one-to-one? Prove it, or provide a counterexample.
   (b) Is f onto? Prove it, or provide a counterexample.

3. Let $A = \{x \in \mathbb{R} : x \neq -1\}$. Consider the function $f\colon A \to \mathbb{R}$ defined by $f(x) = \frac{2x}{x+1}$. Prove that f is one-to-one.
4. Let $A = \{x \in \mathbb{R} : x \neq 2\}$. Prove that the function $f\colon A \to \mathbb{R}$ defined by $f(x) = \frac{4x}{x-2}$ is not onto.
5. Let $A = \{x \in \mathbb{R} : x \neq 2\}$ and let $B = \{y \in \mathbb{R} : y \neq 4\}$. Define the function $f\colon A \to B$ by $f(x) = \frac{4x}{x-2}$. Prove that f is onto.
6. Let $A = \{x \in \mathbb{R} : x \neq 2\}$ and let $B = \{y \in \mathbb{R} : y \neq 4\}$. Prove the function $f\colon A \to B$ defined by $f(x) = \frac{4x}{x-2}$ is one-to-one.
7. Let $a, b \in \mathbb{R}$ with $a \neq 0$ and define the function $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = ax + b$. Given that f is one-to-one and onto, find a formula for the inverse function $f^{-1}\colon \mathbb{R} \to \mathbb{R}$.
8. Suppose that $f\colon \mathbb{R} \to \mathbb{R}^+$ is one-to-one. Define $g\colon \mathbb{R} \to \mathbb{R}^+$ by $g(x) = (f(x))^2$. Prove that g is one-to-one. (Recall that $\sqrt{x^2} = |x|$.)
9. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is one-to-one and let $a, b \in \mathbb{R}$ where $a \neq 0$. Define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) = af(x) + b$. Prove that g is one-to-one.
10. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is onto and let $a, b \in \mathbb{R}$ where $a \neq 0$. Define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) = af(x) + b$. Prove that g is onto.
11. Let $f\colon B \to C$ and $g\colon A \to B$. Suppose that $(f \circ g)\colon A \to C$ is onto. Prove that f is onto.
12. Let $g\colon A \to B$ and $f\colon B \to C$. Suppose that $(f \circ g)\colon A \to C$ is onto and f is one-to-one. Prove that g is onto.
13. Let A, B and C be sets. Prove that if $A \subseteq B$ and $B \cap C = \emptyset$, then $A \subseteq B \setminus C$.
14. Let A, B and C be sets. Prove that if $A \setminus B \subseteq C$ and $A \nsubseteq C$, then $A \cap B \neq \emptyset$.
15. Suppose $g\colon A \to B$ and $f\colon B \to C$ are one-to-one. Prove that $(f \circ g)\colon A \to C$ is one-to-one.
16. Suppose that $g\colon A \to B$ and $f\colon B \to C$ are onto. Prove that $(f \circ g)\colon A \to C$ is onto.
17. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is onto. Let $c \in \mathbb{R}$ be non-zero. Define the function $h\colon \mathbb{R} \to \mathbb{R}$ by $h(x) = cf(x)$. Prove that the function h is onto.
18. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is one-to-one. Let $c \in \mathbb{R}$ be non-zero. Define the function $h\colon \mathbb{R} \to \mathbb{R}$ by $h(x) = cf(x)$. Prove that the function h is one-to-one.
19. Let A and B be sets. Prove that $B \cup (A \setminus B) = A \cup B$.
20. Let A and B be sets. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
21. Let $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ be indexed families of sets with the same indexed set I. Suppose $A_i \subseteq B_i$ for all $i \in I$. Prove that $\bigcup_{i \in I} A_i \subseteq \bigcup_{i \in I} B_i$.
22. Let $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ be indexed families of sets with the same indexed set I. Suppose $A_i \subseteq B_i$ for all $i \in I$. Prove that $\bigcap_{i \in I} A_i \subseteq \bigcap_{i \in I} B_i$.

23. Let $\{A_i : i \in I\}$ and $\{B_j : j \in J\}$ be indexed families of sets. Suppose that there is an $i_0 \in I$ such that $A_{i_0} \subseteq B_j$ for all $j \in J$. Prove that $\bigcap_{i \in I} A_i \subseteq \bigcap_{j \in J} B_j$.

24. Let $\{A_i : i \in I\}$ be an indexed family of sets. Prove that $X \subseteq \bigcap_{i \in I} A_i$ if and only if $X \subseteq A_i$ for all $i \in I$.

25. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is onto. Define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) = (f(x))^2$. Show that g is not one-to-one.

26. Let $a, b \in \mathbb{R}$. Define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) = ax + b$. Show that if g is one-to-one, then $a \neq 0$.

27. Let $g\colon A \to B$ and $f\colon B \to C$. Suppose that $(f \circ g)\colon A \to C$ is one-to-one. Prove that g is one-to-one.

28. Let $g\colon A \to B$ and $f\colon B \to C$. Suppose that $(f \circ g)\colon A \to C$ is one-to-one and that g is onto. Prove that f is one-to-one.

29. Let $f\colon B \to C$ and $g\colon A \to B$. Suppose that $(f \circ g)\colon A \to C$ is onto. Prove that f is onto.

30. Let $g\colon A \to B$ and $f\colon B \to C$. Suppose that $(f \circ g)\colon A \to C$ is onto and that f is one-to-one. Prove that g is onto.

31. Prove the following theorems:
    (a) **Theorem.** Let A be a set and $\{B_i : i \in I\}$ be an indexed family of sets. Then $A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} (A \cap B_i)$.
    (b) **Theorem.** Let A be a set and $\{B_i : i \in I\}$ be an indexed family of sets. Then $A \cup \bigcap_{i \in I} B_i = \bigcap_{i \in I} (A \cup B_i)$.
    (c) **Theorem.** Let A be a set and $\{B_i : i \in I\}$ be an indexed family of sets. Then $A \setminus \bigcap_{i \in I} B_i = \bigcup_{i \in I} (A \setminus B_i)$.

Proof Strategies

To PROVE that $\mathcal{A} \subseteq \mathcal{B}$, use the diagram

Let $x \in \mathcal{A}$.
    Prove $x \in \mathcal{B}$.

To PROVE that two sets $\mathcal{A}$ and $\mathcal{B}$ are equal, use the proof diagram:

Prove $\mathcal{A} \subseteq \mathcal{B}$
Prove $\mathcal{B} \subseteq \mathcal{A}$.

To PROVE that two sets $\mathcal{A}$ and $\mathcal{B}$ are equal, use the proof diagram:

Let x be arbitrary.
    Prove $x \in \mathcal{A} \leftrightarrow x \in \mathcal{B}$.

To PROVE that a function $f\colon A \to B$ is One-To-One, use the proof diagram:

Let $a \in A$ and $b \in A$ be arbitrary.
    Assume $f(a) = f(b)$
        Prove $a = b$.

To PROVE that a function $f\colon A \to B$ is Onto, use the proof diagram:

Let $y \in B$ be arbitrary.
    Let $x =$ (the value you found).
        Prove $f(x) = y$.

Definition. Given two functions $g\colon A \to B$ and $f\colon B \to C$, one forms the *composition function* $(f \circ g)\colon A \to C$ by defining $(f \circ g)(x) = f(g(x))$ for all $x \in A$.

Remark. Let $\{C_i : i \in I\}$ be an indexed family of sets. Then the following statements are true.

(1) $x \in \bigcup_{i \in I} C_i$ iff $x \in C_i$ for some $i \in I$.
(2) $x \notin \bigcup_{i \in I} C_i$ iff $x \notin C_i$ for every $i \in I$.
(3) $x \in \bigcap_{i \in I} C_i$ iff $x \in C_i$ for every $i \in I$.
(4) $x \notin \bigcap_{i \in I} C_i$ iff $x \notin C_i$ for some $i \in I$.