MAT 300 - Final Exam Review Problems on Chapters 5 and 6
Final Exam on Monday Dec. 9, 9:40 to 11:00. Bacon 214A

1. Prove the following theorems:

   (a) **Theorem.** $(A \setminus B) \cap (C \setminus B) = (A \cap C) \setminus B$.

   Proof. Let x be given. We prove $x \in (A \setminus B) \cap (C \setminus B)$ iff $x \in (A \cap C) \setminus B$, as follows:

$$
\begin{aligned}
x \in (A \setminus B) \cap (C \setminus B) &\text{ iff } x \in (A \setminus B) \wedge x \in (C \setminus B) && \text{by the definition of } \cap \\
&\text{ iff } (x \in A \wedge x \notin B) \wedge (x \in C \wedge x \notin B) && \text{by the definition of } \setminus \\
&\text{ iff } (x \in A \wedge x \in C) \wedge (x \notin B \wedge x \notin B) && \text{by comm. \& assoc. logic laws} \\
&\text{ iff } (x \in A \wedge x \in C) \wedge x \notin B && \text{by idempotent logic law} \\
&\text{ iff } (x \in A \cap C) \wedge x \notin B && \text{by the definition of } \cap \\
&\text{ iff } x \in (A \cap C) \setminus B && \text{by the definition of } \setminus.
\end{aligned}
$$

   Therefore, $(A \setminus B) \cap (C \setminus B) = (A \cap C) \setminus B$. □

   (b) **Theorem.** $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$.

   Proof. Let A and B be sets. [We will apply the double subset strategy.]

   ($\subseteq$). Let $x \in (A \cup B) \setminus (A \cap B)$. So $x \in (A \cup B)$ and $x \notin (A \cap B)$. Thus, we have the two items:

$$\text{either } x \in A \text{ or } x \in B, \tag{1}$$
$$\text{either } x \notin A \text{ or } x \notin B. \tag{2}$$

   By (1), we have that either $x \in A$ or $x \in B$.[1] We consider these two cases separately.
   Case 1. If $x \in A$, then (2) implies[2] that $x \notin B$. Hence, $x \in A \setminus B$. Therefore, $x \in (A \setminus B) \cup (B \setminus A)$.
   Case 2. If $x \in B$, then (2) implies that $x \notin A$. Hence, $x \in B \setminus A$. Thus, $x \in (A \setminus B) \cup (B \setminus A)$.

   ($\supseteq$). Let $x \in (A \setminus B) \cup (B \setminus A)$. Thus, either $x \in (A \setminus B)$ or $x \in (B \setminus A)$. We consider these two cases separately.
   Case 1. If $x \in (A \setminus B)$, then $x \in A$ and $x \notin B$. Hence, $x \in (A \cup B)$ and $x \notin (A \cap B)$. Therefore, $x \in (A \cup B) \setminus (A \cap B)$.
   Case 2. If $x \in (B \setminus A)$, then $x \in B$ and $x \notin A$. Hence, $x \in (A \cup B)$ and $x \notin (A \cap B)$. Therefore, $x \in (A \cup B) \setminus (A \cap B)$.
   Therefore, $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$. □

   (c) **Theorem.** If $A \setminus B \subseteq C$, then $A \setminus C \subseteq B$.

   Proof. Assume that (▲) $A \setminus B \subseteq C$. We prove that $A \setminus C \subseteq B$. Let $x \in A \setminus C$. Thus, $x \in A$ and $x \notin C$. We must prove that $x \in B$. Suppose, for a contradiction, that $x \notin B$. Thus, $x \in A$ and $x \notin B$. So $x \in A \setminus B$. So (▲) implies that $x \in C$ and this contradicts to fact that $x \notin C$. Hence, we must have that $x \in B$. Therefore, $A \setminus C \subseteq B$. □

2. Define $f\colon \mathbb{Z} \to \mathbb{Z}$ by $f(n) = 3n$.

   (a) Is f one-to-one? Prove it, or provide a counterexample.
   (b) Is f onto? Prove it, or provide a counterexample.

3. Let $A = \{x \in \mathbb{R} : x \neq -1\}$. Consider the function $f\colon A \to \mathbb{R}$ defined by $f(x) = \frac{2x}{x+1}$. Prove that f is one-to-one.

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[1] We are assuming an 'or' statement. See Assumption Strategy 3.6.3 on page 86 of text.
[2] By Disjunctive Syllogism.

4. Let $A = \{x \in \mathbb{R} : x \neq 2\}$. Prove that the function $f\colon A \to \mathbb{R}$ defined by $f(x) = \frac{4x}{x-2}$ is not onto.
5. Let $A = \{x \in \mathbb{R} : x \neq 2\}$ and let $B = \{y \in \mathbb{R} : y \neq 4\}$. Define the function $f\colon A \to B$ by $f(x) = \frac{4x}{x-2}$. Prove that f is onto.
6. Let $A = \{x \in \mathbb{R} : x \neq 2\}$ and let $B = \{y \in \mathbb{R} : y \neq 4\}$. Prove the function $f\colon A \to B$ defined by $f(x) = \frac{4x}{x-2}$ is one-to-one.
7. Let $a, b \in \mathbb{R}$ with $a \neq 0$ and define the function $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = ax + b$. Given that f is one-to-one and onto, find a formula for the inverse function $f^{-1}\colon \mathbb{R} \to \mathbb{R}$.

   Solution. Solving $ax + b = y$ for x, we obtain $x = \frac{y-b}{a}$. Therefore, $f^{-1}(y) = \frac{y-b}{a}$ is a formula for f^{-1}.

8. Suppose that $f\colon \mathbb{R} \to \mathbb{R}^+$ is one-to-one. Define $g\colon \mathbb{R} \to \mathbb{R}^+$ by $g(x) = (f(x))^2$. Prove that g is one-to-one. (Recall that $\sqrt{x^2} = |x|$.)

   Proof. Assume that $f\colon \mathbb{R} \to \mathbb{R}^+$ is one-to-one. We shall prove that the function $g\colon \mathbb{R} \to \mathbb{R}^+$ defined by $g(x) = (f(x))^2$ is one-to-one. Let $x, y \in \mathbb{R}$. Assume that (1) $g(x) = g(y)$. We shall prove that $x = y$. First, we shall prove that $f(x) = f(y)$ as follows:

$$
\begin{aligned}
g(x) &= g(y) && \text{by (1)} \\
(f(x))^2 &= (f(y))^2 && \text{by the definition of } g \\
\sqrt{(f(x))^2} &= \sqrt{(f(y))^2} && \text{by taking the square root of both sides} \\
|f(x)| &= |f(y)| && \text{because } \sqrt{z^2} = |z| \\
f(x) &= f(y) && \text{because } f(x) > 0 \text{ and } f(y) > 0.
\end{aligned}
$$

   Therefore, $f(x) = f(y)$. Since f is one-to-one, we now conclude that $x = y$. □

9. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is one-to-one and let $a, b \in \mathbb{R}$ where $a \neq 0$. Define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) = af(x) + b$. Prove that g is one-to-one.
10. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is onto and let $a, b \in \mathbb{R}$ where $a \neq 0$. Define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) = af(x) + b$. Prove that g is onto.
11. Let $f\colon B \to C$ and $g\colon A \to B$. Suppose that $(f \circ g)\colon A \to C$ is onto. Prove that f is onto.

   Proof. Let $f\colon B \to C$ and $g\colon A \to B$. Assume $(f \circ g)\colon A \to C$ is onto. We shall prove $f\colon B \to C$ is onto. Let $y \in C$. Since $(f \circ g)\colon A \to C$ is onto and $y \in C$, there is an $a \in A$ such that (1) $(f \circ g)(a) = y$. Let $x = g(a)$. Clearly, $x \in B$. We shall prove $f(x) = y$ as follows:

$$
\begin{aligned}
f(x) &= f(g(a)) && \text{because } x = g(a) \\
&= (f \circ g)(a) && \text{by the definition of composition} \\
&= y && \text{by (1).}
\end{aligned}
$$

   Therefore, $f(x) = y$ and so, f is onto. □

12. Let $g\colon A \to B$ and $f\colon B \to C$. Suppose that $(f \circ g)\colon A \to C$ is onto and f is one-to-one. Prove that g is onto.

   Proof. Let $g\colon A \to B$ and $f\colon B \to C$. Assume $(f \circ g)\colon A \to C$ is onto and f is one-to-one. We shall prove $g\colon A \to B$ is onto. Let $y \in B$. Clearly, $f(y) \in C$. Since $(f \circ g)\colon A \to C$ is onto and $f(y) \in C$, there is an $a \in A$ such that $(f \circ g)(a) = f(y)$. We shall prove that $g(a) = y$, Since $(f \circ g)(a) = f(y)$, we see that

$$f(g(a)) = f(y).$$

   Because f is one-to-one, we conclude that $g(a) = y$ and thus, g is onto. □

13. Let A, B and C be sets. Prove that if $A \subseteq B$ and $B \cap C = \emptyset$, then $A \subseteq B \setminus C$.

Proof. Let A, B and C be sets. Assume that

$$A \subseteq B \tag{1}$$
$$B \cap C = \emptyset. \tag{2}$$

We shall prove that $A \subseteq B \setminus C$. Let $x \in A$. Thus, $x \in B$ by (1). Since $B \cap C = \emptyset$, we conclude that $x \notin C$. Since $x \in B$ and $x \notin C$, we have that $x \in B \setminus C$. □

14. Let A, B and C be sets. Prove that if $A \setminus B \subseteq C$ and $A \not\subseteq C$, then $A \cap B \neq \emptyset$.

Proof. Let A, B and C be sets. Assume that

$$A \setminus B \subseteq C \tag{1}$$
$$A \not\subseteq C. \tag{2}$$

We shall prove that $A \cap B \neq \emptyset$. Since $A \not\subseteq C$, there is an $x \in A$ such that $x \notin C$. We shall prove that $x \in B$. Suppose, for a contradiction, that $x \notin B$. Then $x \in A \setminus B$. By (1) we have that $x \in C$ which contradicts the fact that $x \notin C$. Hence, $x \in B$. Because $x \in A$ and $x \in B$, we see that $x \in A \cap B$. Therefore, $A \cap B \neq \emptyset$. □

15. Suppose $g\colon A \to B$ and $f\colon B \to C$ are one-to-one. Prove that $(f \circ g)\colon A \to C$ is one-to-one.

16. Suppose that $g\colon A \to B$ and $f\colon B \to C$ are onto. Prove that $(f \circ g)\colon A \to C$ is onto.

Proof. Assume $g\colon A \to B$ and $f\colon B \to C$ are onto. We prove that the function $(f \circ g)\colon A \to C$ is onto. Let $z \in C$. Since $f\colon B \to C$ is onto and $z \in C$, there is a $y \in B$ such that $f(y) = z$. Because $y \in B$ and $g\colon A \to B$ is onto, there is an $x \in A$ such that $g(x) = y$. We will show that $(f \circ g)(x) = z$ as follows:

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) && \text{by definition of composition} \\ &= f(y) && \text{because } g(x) = y \\ &= z && \text{because } f(y) = z. \end{aligned}$$

Therefore, $f \circ g\colon A \to C$ is onto. □

17. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is onto. Let $c \in \mathbb{R}$ be non-zero. Define the function $h\colon \mathbb{R} \to \mathbb{R}$ by $h(x) = cf(x)$. Prove that the function h is onto.

Proof. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is onto and $c \neq 0$. Let $h\colon \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = cf(x)$. We shall prove that h is onto. Let $y \in R$. Since $\frac{y}{c} \in \mathbb{R}$ and f is onto, there is an $x \in \mathbb{R}$ such that $(\star)$ $f(x) = \frac{y}{c}$. We prove that $h(x) = y$ as follows:

$$\begin{aligned} h(x) &= cf(x) && \text{by the definition of } h \\ &= c \cdot \frac{y}{c} && \text{by } (\star) \\ &= y && \text{by algebra.} \end{aligned}$$

Therefore, $h(x) = y$ and h is onto. □

18. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is one-to-one. Let $c \in \mathbb{R}$ be non-zero. Define the function $h\colon \mathbb{R} \to \mathbb{R}$ by $h(x) = cf(x)$. Prove that the function h is one-to-one.

Proof. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is one-to-one and $c \neq 0$. Let $h\colon \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = cf(x)$. We shall prove that h is one-to-one. Let $x, y \in R$. Assume that $h(x) = h(y)$. We prove that $x = y$ as follows:

$$\begin{aligned} h(x) = h(y) &\implies cf(x) = cf(y) && \text{by the definition of } h \\ &\implies f(x) = f(y) && \text{because } c \neq 0 \\ &\implies x = y && \text{since } f \text{ is one-to-one.} \end{aligned}$$

Therefore, h is one-to-one. □

19. Let A and B be sets. Prove that $B \cup (A \setminus B) = A \cup B$.

Proof. Let A and B be sets. Let x be given. We prove $x \in B \cup (A \setminus B)$ iff $x \in A \cup B$, as follows:

$$\begin{aligned} x \in B \cup (A \setminus B) &\text{ iff } x \in B \vee x \in (A \setminus B) && \text{by the definition of } \cap \\ &\text{ iff } x \in B \vee (x \in A \wedge x \notin B) && \text{by the definition of } \setminus \\ &\text{ iff } (x \in B \vee x \in A) \wedge (x \in B \vee x \notin B) && \text{by logical distribution} \\ &\text{ iff } x \in B \vee x \in A && \text{by the tautology law} \\ &\text{ iff } x \in A \vee x \in B && \text{by the logical commutativity} \\ &\text{ iff } x \in A \cup B && \text{by the definition of } \cup. \end{aligned}$$

Therefore, $B \cup (A \setminus B) = A \cup B$. □

20. Let A and B be sets. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. Let x be given. We prove that $x \in A \cap (B \cup C)$ if and only if $x \in (A \cap B) \cup (A \cap C)$, as follows:

$$\begin{aligned} x \in A \cap (B \cup C) &\text{ iff } x \in A \wedge x \in (B \cup C) && \text{by the definition of } \cap \\ &\text{ iff } x \in A \wedge (x \in B \vee x \in C) && \text{by the definition of } \cup \\ &\text{ iff } (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) && \text{by logical distributivity} \\ &\text{ iff } x \in A \cap B \vee x \in A \cap C && \text{by the definition of } \cap \\ &\text{ iff } x \in (A \cap B) \cup (A \cap C) && \text{by the definition of } \cup. \end{aligned}$$

Therefore, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. □

21. Let $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ be indexed families of sets with the same indexed set I. Suppose $A_i \subseteq B_i$ for all $i \in I$. Prove that $\bigcup_{i \in I} A_i \subseteq \bigcup_{i \in I} B_i$.
22. Let $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ be indexed families of sets with the same indexed set I. Suppose $A_i \subseteq B_i$ for all $i \in I$. Prove that $\bigcap_{i \in I} A_i \subseteq \bigcap_{i \in I} B_i$.
23. Let $\{A_i : i \in I\}$ and $\{B_j : j \in J\}$ be indexed families of sets. Suppose that there is an $i_0 \in I$ such that $A_{i_0} \subseteq B_j$ for all $j \in J$. Prove that $\bigcap_{i \in I} A_i \subseteq \bigcap_{j \in J} B_j$.
24. Let $\{A_i : i \in I\}$ be an indexed family of sets. Prove that $X \subseteq \bigcap_{i \in I} A_i$ if and only if $X \subseteq A_i$ for all $i \in I$.
25. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is onto. Define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) = (f(x))^2$. Show that g is not one-to-one.
26. Let $a, b \in \mathbb{R}$. Define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) = ax + b$. Show that if g is one-to-one, then $a \neq 0$.
27. Let $g\colon A \to B$ and $f\colon B \to C$. Suppose that $(f \circ g)\colon A \to C$ is one-to-one. Prove that g is one-to-one.
28. Let $g\colon A \to B$ and $f\colon B \to C$. Suppose that $(f \circ g)\colon A \to C$ is one-to-one and that g is onto. Prove that f is one-to-one.

Proof. Let $g\colon A \to B$ and $f\colon B \to C$. Assume that $(f \circ g)\colon A \to C$ is one-to-one and g is onto. To prove that f is one-to-one, let $x \in B$ and $y \in B$. Assume that $f(x) = f(y)$. Since $g\colon A \to B$ is onto, there is a $c \in A$ and $d \in A$ such that $g(c) = x$ and $g(d) = y$. Therefore,

$$f(g(c)) = f(x) \text{ and } f(g(d)) = f(y);$$

That is, by the definition of composition, we have that

$$(f \circ g)(c) = f(x) \text{ and } (f \circ g)(d) = f(y).$$

Since $f(x) = f(y)$, we conclude that

$$(f \circ g)(c) = (f \circ g)(d).$$

Because, $(f \circ g)\colon A \to C$ is one-to-one, we infer that $c = d$. Therefore, $g(c) = g(d)$. As $g(c) = x$ and $g(d) = y$, we have that $x = y$. Therefore, f is one-to-one. □

29. Let $f\colon B \to C$ and $g\colon A \to B$. Suppose that $(f \circ g)\colon A \to C$ is onto. Prove that f is onto.
30. Let $g\colon A \to B$ and $f\colon B \to C$. Suppose that $(f \circ g)\colon A \to C$ is onto and that f is one-to-one. Prove that g is onto.
31. Prove the following theorems:

(a) **Theorem.** Let A be a set and $\{B_i : i \in I\}$ be an indexed family of sets. Then $A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I}(A \cap B_i)$.

Proof. Suppose that A is a set and that $\{B_i : i \in I\}$ is an indexed family of sets. Let x be given. We shall prove that $x \in A \cap \bigcup_{i \in I} B_i$ if and only if $x \in \bigcup_{i \in I}(A \cap B_i)$ as follows:

$$\begin{aligned} x \in A \cap \bigcup_{i \in I} B_i &\text{ iff } x \in A \text{ and } x \in \bigcup_{i \in I} B_i && \text{by the definition of } \cap \\ &\text{ iff } x \in A \text{ and } x \in B_i \text{ for some } i \in I && \text{by the definition of } \bigcup \\ &\text{ iff } x \in A \cap B_i \text{ for some } i \in I && \text{by the definition of } \cap \\ &\text{ iff } x \in \bigcup_{i \in I}(A \cap B_i) && \text{by the definition of } \textstyle\bigcup. \end{aligned}$$

Therefore, $A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I}(A \cap B_i)$. □

(b) **Theorem.** Let A be a set and $\{B_i : i \in I\}$ be an indexed family of sets. Then $A \cup \bigcap_{i \in I} B_i = \bigcap_{i \in I}(A \cup B_i)$.

Proof. Suppose that A is a set and that $\{B_i : i \in I\}$ is an indexed family of sets. Let x be given. We shall prove that $x \in A \cup \bigcap_{i \in I} B_i$ if and only if $x \in \bigcap_{i \in I}(A \cup B_i)$ as follows:

$$\begin{aligned} x \in A \cup \bigcap_{i \in I} B_i &\text{ iff } x \in A \text{ or } x \in \bigcap_{i \in I} B_i && \text{by the definition of } \cup \\ &\text{ iff } x \in A \text{ or } x \in B_i \text{ for all } i \in I && \text{by the definition of } \bigcap \\ &\text{ iff } x \in A \cup B_i \text{ for all } i \in I && \text{by the definition of } \cup \\ &\text{ iff } x \in \bigcap_{i \in I}(A \cup B_i) && \text{by the definition of } \textstyle\bigcap. \end{aligned}$$

Therefore, $A \cup \bigcap_{i \in I} B_i = \bigcap_{i \in I}(A \cup B_i)$. □

(c) **Theorem.** Let A be a set and $\{B_i : i \in I\}$ be an indexed family of sets. Then $A \setminus \bigcap_{i \in I} B_i = \bigcup_{i \in I}(A \setminus B_i)$.

Proof. Suppose that A is a set and that $\{B_i : i \in I\}$ is an indexed family of sets. Let x be given. We shall prove that $x \in A \setminus \bigcap_{i\in I} B_i$ if and only if $x \in \bigcup_{i\in I}(A \setminus B_i)$ as follows:

$$\begin{aligned}
x \in A \setminus \bigcap_{i\in I} B_i &\text{ iff } x \in A \text{ and } x \notin \bigcap_{i\in I} B_i && \text{by the definition of } \setminus \\
&\text{iff } x \in A \text{ and } x \notin B_i \text{ for some } i \in I && \text{by the definition of } \bigcap \\
&\text{iff } x \in A \setminus B_i \text{ for some } i \in I && \text{by the definition of } \setminus \\
&\text{iff } x \in \bigcup_{i\in I}(A \setminus B_i) && \text{by the definition of } \bigcup.
\end{aligned}$$

Therefore, $A \setminus \bigcap_{i\in I} B_i = \bigcup_{i\in I}(A \setminus B_i)$. □

Proof Strategies

To PROVE that $\mathcal{A} \subseteq \mathcal{B}$, use the diagram

Let $x \in \mathcal{A}$.
  Prove $x \in \mathcal{B}$.

To PROVE that two sets $\mathcal{A}$ **and** $\mathcal{B}$ **are equal**, use the proof diagram:

Prove $\mathcal{A} \subseteq \mathcal{B}$
Prove $\mathcal{B} \subseteq \mathcal{A}$.

To PROVE that two sets $\mathcal{A}$ **and** $\mathcal{B}$ **are equal**, use the proof diagram:

Let x be arbitrary.
  Prove $x \in \mathcal{A} \leftrightarrow x \in \mathcal{B}$.

To PROVE that a function $f\colon A \to B$ **is One-To-One**, use the proof diagram:

Let $a \in A$ and $b \in A$ be arbitrary.
  Assume $f(a) = f(b)$
    Prove $a = b$.

To PROVE that a function $f\colon A \to B$ **is Onto**, use the proof diagram:

Let $y \in B$ be arbitrary.
  Let $x =$ (the value you found).
    Prove $f(x) = y$.

Definition. Given two functions $g\colon A \to B$ and $f\colon B \to C$, one forms the *composition function* $(f \circ g)\colon A \to C$ by defining $(f \circ g)(x) = f(g(x))$ for all $x \in A$.

Remark. Let $\{C_i : i \in I\}$ be an indexed family of sets. Then the following statements are true.

(1) $x \in \bigcup_{i\in I} C_i$ iff $x \in C_i$ for some $i \in I$.

(2) $x \notin \bigcup_{i\in I} C_i$ iff $x \notin C_i$ for every $i \in I$.

(3) $x \in \bigcap_{i\in I} C_i$ iff $x \in C_i$ for every $i \in I$.

(4) $x \notin \bigcap_{i\in I} C_i$ iff $x \notin C_i$ for some $i \in I$.