MAT 300 Review Problems for Chapter 3 and Sections 4.2, 4.4
Exam #2 on Friday, November 8, 2019

A proof by mathematical induction must NOT have the notations $P(1)$, $P(n)$, or $P(n{+}1)$ appearing anywhere in the proof.

1. Theorem. Let x and y be real numbers. Then $(x-y)(x^2+xy+y^2)=x^3-y^3$.

Proof. Let x and y be real numbers. We prove that $(x-y)(x^2+xy+y^2)=x^3-y^3$ as follows:

$$\begin{aligned}(x-y)(x^2+xy+y^2) &= x^3+x^2y+xy^2-x^2y-xy^2-y^3 && \text{by the distribution property}\\ &= x^3-y^3 && \text{by algebra.}\end{aligned}$$

Thus, we have that $(x-y)(x^2+xy+y^2)=x^3-y^3$. □

2. Theorem. Let a and b be real numbers. If $a<0$ and $b<0$, then $(a+b)^2>a^2+b^2$.

Proof. Let a and b be real numbers. Assume that $a<0$ and $b<0$. Thus, $ab>0$ by property of inequalities. Thus, (1) $2ab>0$ also by property of inequalities. We now show that $(a+b)^2>a^2+b^2$ as follows:

$$\begin{aligned}(a+b)^2 &= a^2+2ab+b^2 && \text{by algebra}\\ &> a^2+0+b^2 && \text{by (1) and property of inequality}\\ &= a^2+b^2 && \text{by algebra.}\end{aligned}$$

Therefore, $(a+b)^2>a^2+b^2$. □

3. Theorem. Let $n\geq 2$ be a natural number. If $2^n>n$, then $2^{n+1}>n+1$.

Proof. Let $n\geq 2$ be a natural number. Assume that (1) $2^n>n$. We now show that $2^{n+1}>n+1$ as follows:

$$\begin{aligned}2^{n+1} &= 2\cdot 2^n && \text{by algebra}\\ &> 2\cdot n && \text{by (1) and property of inequality}\\ &= n+n && \text{by algebra}\\ &> n+1 && \text{because } n\geq 2 \text{ implies that } n>1.\end{aligned}$$

Therefore, $2^{n+1}>n+1$. □

4. Theorem. For all real numbers $y>0$, there is a real number $x<0$ such that $y^2+2xy=-x^2$.

Proof. Let $y>0$. Let $x=-y$. Since $y>0$, we see that $x<0$. Since the equation $y^2+2xy=-x^2$ is equivalent to the equation $y^2+2xy+x^2=0$, we prove that $y^2+2xy+x^2=0$, when $x=-y$, as follows:

$$\begin{aligned}y^2+2xy+x^2 &= (y+x)^2 && \text{by algebra}\\ &= (y-y)^2 && \text{because } x=-y\\ &= 0 && \text{by algebra.}\end{aligned}$$

Therefore, $y^2+2xy=-x^2$ when $x=-y$. □

5. Theorem. Let a and b are real numbers such that $a > 0$ and $b < -4$. Then $ab + b < -4(a + 1)$.

Proof. Let a and b are real numbers. Assume $a > 0$ and

$$b < -4. \tag{1}$$

Since a is positive, (1) implies that

$$ab < -4a. \tag{2}$$

Inequalities (1) and (2) imply that

$$ab + b < -4a + -4.$$

Thus, by algebra, we conclude that $ab + b < -4(a + 1)$. □

6. Theorem. Suppose n is an integer. Then $15 \mid n$ if and only if $3 \mid n$ and $5 \mid n$.

Proof. Let n be an integer. We will prove that $15 \mid n$ if and only if $3 \mid n$ and $5 \mid n$.

($\Rightarrow$). Assume $15 \mid n$. Thus, $n = 15i$ for some $i \in \mathbb{Z}$. So, $n = 15i = 3(5i) = 5(3i)$ and hence, $3 \mid n$ and $5 \mid n$.

($\Leftarrow$). Assume $3 \mid n$ and $5 \mid n$. Thus there are integers i and j such that $n = 3i$ and $n = 5j$. Therefore,

$$n = 10n - 9n = 10(3i) - 9(5j) = 15(2i) - 15(3j) = 15(2i - 3j).$$

Hence, $n = 15k$ where $k = 2i - 3j$ is an integer. Therefore, $15 \mid n$. □

7. Theorem. Let m, a, b, c, d be integers where $m > 1$. If $m \mid (a - b)$ and $m \mid (c - d)$, then $m \mid ((a + c) - (b + d))$.

Proof. Let m, a, b, c, d be integers where $m > 1$. Assume $m \mid (a - b)$ and $m \mid (c - d)$. Thus,

$$a - b = mi \tag{1}$$
$$c - d = mj \tag{2}$$

for some integers i and j. By adding corresponding sides of equations (1) and (2), we obtain

$$(a - b) + (c - d) = mi + mj.$$

After performing some algebra, we conclude that

$$(a + c) - (b + d) = m(i + j).$$

Therefore, $m \mid ((a + c) - (b + d))$. □

8. Theorem. Let a, b, d be in $\mathbb{R}$. If $0 \le a < d$ and $0 \le b < d$, then $a - b < d$ and $b - a < d$.

Proof. Let a, b, d be real numbers. Assume that (1) $0 \le a < d$ and (2) $0 \le b < d$. We shall prove that $a - b < d$ and $b - a < d$.

First we prove that $a - b < d$. Since $0 \le b$, we see that $-b \le 0$. Therefore, $a - b \le a$. Because $a < d$, we infer that $a - b < d$.

Now we prove that $b - a < d$. Since $0 \le a$, we see that $-a \le 0$. Therefore, $b - a \le b$. Because $b < d$, we infer that $b - a < d$. □

9. Theorem. For all integers a, b, and c, if $c \mid a$ and $c \mid b$, then $c \mid (a - b)$.

10. Theorem. For every $x \in \mathbb{R}$, there is a real number y such that $yx^2 - 3x = -2y$.

Proof. Let x be a real number. Let $y = \frac{3x}{x^2+2}$. Since $x^2 + 2 \neq 0$, we see that y is a real number. Since the equation $yx^2 - 3x = -2y$ is equivalent to the equation $yx^2 - 3x + 2y = 0$, we just need to show that $yx^2 - 3x + 2y = 0$ holds when $y = \frac{3x}{x^2+2}$. We do this as follows:

$$\begin{aligned} yx^2 - 3x + 2y &= y(x^2+2) - 3x && \text{by algebra} \\ &= \left(\frac{3x}{x^2+2}\right)(x^2+2) - 3x && \text{because } y = \frac{3x}{x^2+2} \\ &= 0 && \text{by algebra.} \end{aligned}$$

Therefore, $yx^2 - 3x = -2y$ when $y = \frac{3x}{x^2+2}$. □

11. Theorem. There is a $y \in \mathbb{R}$, such that $yx + 6 = 2x + 3y$ for all real numbers x.

Proof. Let $y = 2$. We now prove that $yx + 6 = 2x + 3y$ for all real numbers x. Let x be any real number. Since $y = 2$, we conclude that $yx+6 = 2x+6$ and $2x+3y = 2x+6$. Thus, $yx+6 = 2x+3y$ for all real numbers x when $y = 2$. □

12. Theorem. For every natural number $n \geq 1$, $2 + 6 + 18 + \cdots + 2 \cdot 3^{n-1} = 3^n - 1$.

Proof. We prove, by mathematical induction, that $2+6+18+\cdots+2\cdot 3^{n-1} = 3^n - 1$ for all $n \geq 1$.

Base step: For $n = 1$, we see that $2 \cdot 3^{1-1} = 3^1 - 1$.

Inductive step: Let $n \geq 1$ and assume the induction hypothesis that

$$2 + 6 + 18 + \cdots + 2 \cdot 3^{n-1} = 3^n - 1. \tag{IH}$$

We show that $2 + 6 + 18 + \cdots + 2 \cdot 3^n = 3^{n+1} - 1$ as follows:

$$\begin{aligned} 2 + 6 + 18 + \cdots + 2 \cdot 3^n &= (2 + 6 + 18 + \cdots + 2 \cdot 3^{n-1}) + 2 \cdot 3^n && \text{by algebra} \\ &= 3^n - 1 + 2 \cdot 3^n && \text{by (IH)} \\ &= 3^n + 2 \cdot 3^n - 1 && \text{by algebra} \\ &= (1+2)3^n - 1 && \text{by algebra} \\ &= 3 \cdot 3^n - 1 && \text{by arithmetic} \\ &= 3^{n+1} - 1 && \text{by algebra.} \end{aligned}$$

Hence, $2 + 6 + 18 + \cdots + 2 \cdot 3^n = 3^{n+1} - 1$ and the proof is complete. □

13. Theorem. Let x and y be positive real numbers. Then $\frac{x+y}{2} \geq \sqrt{xy}$.

Proof. Let x and y be positive real numbers. Assume, for a contradiction, that $\frac{x+y}{2} < \sqrt{xy}$. Thus, $x + y < 2\sqrt{xy}$. Since $x > 0$ and $y > 0$, we see that $x + y$ and $2\sqrt{xy}$ are positive. Thus, by Theorem 3.3.2 of text, we have that $(x+y)^2 < (2\sqrt{xy})^2$. Hence,

$$x^2 + 2xy + y^2 < 4xy$$

and so,

$$x^2 - 2xy + y^2 < 0.$$

Because $(x-y)^2 = x^2 - 2xy + y^2$, we conclude that $(x-y)^2 < 0$ which is a contradiction (see Exercise 8 of Section 3.6). Therefore, $\frac{x+y}{2} \geq \sqrt{xy}$. □

14. Theorem. The real number $2 + \frac{1}{2}\sqrt{2}$ is irrational.

Proof. Suppose, for a contradiction, that $2 + \frac{1}{2}\sqrt{2}$ is rational. Thus,

$$2 + \frac{1}{2}\sqrt{2} = \frac{i}{j} \tag{1}$$

for some integers i and j where $j \neq 0$. Solving the equation (1) for $\sqrt{2}$ and getting common divisors, we obtain

$$\sqrt{2} = \frac{2i - 4j}{2j}.$$

Since $2i - 4j$ and $2j$ are integers where $2j \neq 0$, we conclude that $\sqrt{2}$ is rational which is a contradiction (see Theorem 3.8.8 of text). Therefore, $2 + \frac{1}{2}\sqrt{2}$ is irrational. □

15. Theorem. For every natural number $n \geq 1$, we have $\sum_{k=1}^{n} \frac{1}{4k^2-1} = \frac{n}{2n+1}$.

Proof. We prove, by mathematical induction, that $\sum_{k=1}^{n} \frac{1}{4k^2-1} = \frac{n}{2n+1}$ for all $n \geq 1$.

Base step: For $n = 1$, we see that $\sum_{k=1}^{1} \frac{1}{4k^2-1} = \frac{1}{3}$ and $\frac{1}{2\cdot1+1} = \frac{1}{3}$. Thus, $\sum_{k=1}^{1} \frac{1}{4k^2-1} = \frac{1}{2\cdot1+1}$.

Inductive step: Let $n \geq 1$ be arbitrary and assume the induction hypothesis that

$$\sum_{k=1}^{n} \frac{1}{4k^2 - 1} = \frac{n}{2n+1}. \tag{IH}$$

We show that $\sum_{k=1}^{n+1} \frac{1}{4k^2-1} = \frac{n+1}{2n+3}$ as follows:

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{1}{4k^2 - 1} &= \left(\sum_{k=1}^{n} \frac{1}{4k^2 - 1}\right) + \frac{1}{4(n+1)^2 - 1} && \text{by regrouping} \\
&= \frac{n}{2n+1} + \frac{1}{4(n+1)^2 - 1} && \text{by (IH)} \\
&= \frac{n}{2n+1} + \frac{1}{(2(n+1)+1)(2(n+1)-1)} && \text{by factoring} \\
&= \frac{n}{2n+1} + \frac{1}{(2n+3)(2n+1)} && \text{by algebra} \\
&= \frac{1}{2n+1}\left[n + \frac{1}{2n+3}\right] && \text{by algebra} \\
&= \frac{1}{2n+1}\left[\frac{2n^2+3n+1}{2n+3}\right] && \text{by common denominator} \\
&= \frac{1}{2n+1}\left[\frac{(2n+1)(n+1)}{2n+3}\right] && \text{by factoring} \\
&= \frac{n+1}{2n+3} && \text{by algebra.}
\end{aligned}$$

Hence, $\sum_{k=1}^{n+1} \frac{1}{4k^2-1} = \frac{n+1}{2n+3}$ and the proof is complete. □

16. Theorem. $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$, for all natural numbers $n \geq 1$.

Proof. We prove, by mathematical induction, that $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$ for all $n \geq 1$.

Base step: For $n = 1$, we see that $\sum_{k=1}^{1} k \cdot k! = 1 \cdot ! = 1$ and $(1+1)! - 1 = 2! - 1 = 1$. Thus, $\sum_{k=1}^{1} k \cdot k! = (1+1)! - 1$.

Inductive step: Let $n \geq 1$ be arbitrary and assume the induction hypothesis that

$$\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1. \tag{IH}$$

We show that $\sum_{k=1}^{n+1} k \cdot k! = (n+2)! - 1$ as follows:

$$\begin{aligned}
\sum_{k=1}^{n+1} k \cdot k! &= \left(\sum_{k=1}^{n} k \cdot k!\right) + (n+1) \cdot (n+1)! && \text{by regrouping} \\
&= (n+1)! - 1 + (n+1) \cdot (n+1)! && \text{by (IH)} \\
&= (n+1)! + (n+1) \cdot (n+1)! - 1 && \text{by algebra} \\
&= (1+n+1)(n+1)! - 1 && \text{by algebra} \\
&= (n+2)(n+1)! - 1 && \text{by algebra} \\
&= (n+2)! - 1 && \text{because } (n+2)(n+1)! = (n+2)!.
\end{aligned}$$

Hence, $\sum_{k=1}^{n+1} k \cdot k! = (n+2)! - 1$ and the proof is complete. □

17. Theorem. $8 \mid (9^n - 1)$, for every integer $n \geq 1$.

Proof. We prove, by mathematical induction, that $8 \mid (9^n - 1)$ for all $n \geq 1$.

Base step: For $n = 1$, we see that $9^1 - 1 = 8$. Thus, $8 \mid (9^1 - 1)$.

Inductive step: Let $n \geq 1$ and assume the induction hypothesis that $8 \mid (9^n - 1)$; that is, assume

$$9^n - 1 = 8i \tag{IH}$$

for some integer i. We show that $8 \mid (9^{n+1} - 1)$ as follows:

$$\begin{aligned}
9^{n+1} - 1 &= 9 \cdot 9^n - 1 && \text{by algebra} \\
&= (8+1)9^n - 1 && \text{as } 8+1 = 9 \\
&= 8 \cdot 9^n + (9^n - 1) && \text{by algebra} \\
&= 8 \cdot 9^n + 8i && \text{by (IH)} \\
&= 8(9^n + i) && \text{by algebra.}
\end{aligned}$$

So $9^{n+1} - 1 = 8(9^n + i)$ where $9^n + i$ is an integer. Thus, $8 \mid (9^{n+1} - 1)$. □

18. Theorem. Let $x \geq -1$ be a real number. Then $(1+x)^n \geq 1+nx$ for all integers $n \geq 1$.

Proof. Let $x \geq -1$ be a real number. Thus, $(\star)$ $1+x \geq 0$. We prove, by mathematical induction, that $(1+x)^n \geq 1+nx$ for all $n \geq 1$.

Base step: For $n=1$, we see that $(1+x)^1 = 1+x$ and $1+1\cdot x = 1+x$. Thus, $(1+x)^1 \geq 1+1\cdot x$.

Inductive step: Let $n \geq 1$ and assume the induction hypothesis that

$$(1+x)^n \geq 1+nx. \tag{IH}$$

We show that $(1+x)^{n+1} \geq 1+(n+1)x$ as follows:

$$\begin{aligned}
(1+x)^{n+1} &= (1+x)(1+x)^n && \text{by algebra}\\
&\geq (1+x)(1+nx) && \text{by (IH), } (\star)\text{, and prop. of inequality}\\
&= 1+nx+x+nx^2 && \text{by algebra}\\
&= 1+(n+1)x+nx^2 && \text{by algebra}\\
&\geq 1+(n+1)x && \text{as } nx^2 \geq 0.
\end{aligned}$$

Hence, $(1+x)^{n+1} \geq 1+(n+1)x$ and the proof is complete. □

19. Theorem. For every integer $n \geq 1$, $\sum_{k=1}^{n} 2\cdot 3^{k-1} = 3^n - 1$.

Proof. We prove, by mathematical induction, that $\sum_{k=1}^{n} 2\cdot 3^{k-1} = 3^n - 1$ for all $n \geq 1$.

Base step: For $n=1$, we see that $\sum_{k=1}^{1} 2\cdot 3^{k-1} = 2\cdot 3^{1-1} = 2$ and $3^1-1 = 2$. Thus, $\sum_{k=1}^{1} 2\cdot 3^{k-1} = 3^1-1$.

Inductive step: Let $n \geq 1$ be arbitrary and assume the induction hypothesis that

$$\sum_{k=1}^{n} 2\cdot 3^{k-1} = 3^n - 1. \tag{IH}$$

We show that $\sum_{k=1}^{n+1} 2\cdot 3^{k-1} = 3^{n+1} - 1$ as follows:

$$\begin{aligned}
\sum_{k=1}^{n+1} 2\cdot 3^{k-1} &= \left(\sum_{k=1}^{n} 2\cdot 3^{k-1}\right) + 2\cdot 3^{n+1-1} && \text{by regrouping}\\
&= 3^n - 1 + 2\cdot 3^{n+1-1} && \text{by (IH)}\\
&= 3^n - 1 + 2\cdot 3^n && \text{by algebra}\\
&= 3^n + 2\cdot 3^n - 1 && \text{by algebra}\\
&= (1+2)3^n - 1 && \text{by algebra}\\
&= 3\cdot 3^n - 1 && \text{by aritmetic}\\
&= 3^{n+1} - 1 && \text{by algebra.}
\end{aligned}$$

Hence, $\sum_{k=1}^{n+1} 2\cdot 3^{k-1} = 3^{n+1} - 1$ and the proof is complete. □