CHAPTER 1

Propositional Logic

A mathematician establishes the truth of a mathematical statement by providing a proof. Such a proof often uses principles of reasoning that are best described within propositional logic. In this chapter we examine the basic tools in propositional logic that mathematicians use to demonstrate that their conclusions are valid. What is a proposition? A *proposition* is a declarative sentence or assertion that is either true or false. Here are two such propositions:

(1) Global warming is a serious problem.
(2) Paris is in France.

Propositional logic studies the results of combining propositions to form more complex statements. In particular, the following three sentences each contain the above two propositions (1) and (2) as components.

Global warming is a serious problem and Paris is in France.
Global warming is a serious problem or Paris is in France.
If Paris is in France, then global warming is not a serious problem.

We will pursue the meaning of assertions that are obtained by connecting statements using "and," "or," "not," "if–then," and "if and only if." These five expressions are frequently used in mathematics. For example, consider the following three mathematical statements:

$x \geq 2$ or $x \leq 2$,
if $x \geq 3$, then $x > 1$,
$x \geq 2$ if and only if $x + 5 \geq 7$,

where x is a real number. Are these three statements true for all real numbers x? If so, then they must be true for $x = 3$, $x = 2$, and for $x = 1$. We will address such issues here in this chapter.

1.1 Logical Form and Logical Equivalence

Symbols play a critical role in mathematics and science. When discussing the logic of propositional statements, we shall use symbols to represent these statements. Capital letters, for instance P, Q, R, will stand for propositional statements, or *propositional components*. As an example, we could use the letter G to represent the component "Global warming is a serious problem" and use the letter P to denote the component "Paris is in France."

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We shall identify symbols for each of the English connectors "and," "or," "not." We will use the symbol $\wedge$ to represent "and," the symbol $\vee$ to represent "or," and use the symbol $\neg$ to represent "not." In addition, the symbol $\rightarrow$ denotes the word "implies" and the symbol $\leftrightarrow$ represents the phrase "if and only if." The five symbols $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$ are called *logical connectives*. Using these logical connectives, we will be able to analyze the logical structure of an English sentence. For instance, the logical form of the sentence "Global warming is a serious problem and Paris is in France" can now be expressed by $G \wedge P$.

In this section we shall focus on the logical meaning of $\wedge, \vee, \neg$. We will then explore the connectives $\rightarrow$ and $\leftrightarrow$ in Section 1.2. We shall refer to the three logical connectives $\wedge, \vee, \neg$ as *conjunction, disjunction*, and *negation*, respectively. Given a list of propositional components $A, B, C, \ldots$, and the logical connectives $\wedge, \vee, \neg$, we can form *propositional sentences*. For example,

1. $P \wedge Q$ (means "P and Q").
2. $P \vee Q$ (means "P or Q").
3. $\neg P$ (means "not P" or "it is not the case that P").

Using propositional components as building blocks and the connectives as mortar, one can construct more complicated propositional sentences, for example,

$$(P \wedge \neg Q) \vee (\neg S \wedge R).$$

It is important to use parentheses so that our propositional sentences are clear and readable; however, we shall use the two conventions:

1. The *outermost* parentheses need not be explicitly written. Thus, we can write $A \wedge B$ to denote $(A \wedge B)$.
2. The negation symbol shall apply to as little as possible. We can therefore write $\neg A \wedge B$ to denote $(\neg A) \wedge B$.

1.1.1 Analyzing the Logical Form of English Statements

Virtually every English statement can be expressed as a propositional sentence. All one has to do is first identify the propositional components that appear in the English sentence and then identify the logical connectives that also appear in the sentence. There are times when these logical connectives are not explicitly stated and may be somewhat hidden by the words used in the sentence. We will see sentences that contain "hidden connectives" in our next example.

Example 1. Analyze the logical form of the following seven English statements. In other words write each statement symbolically and thereby reveal any hidden logical connectives.

1. Emily writes poetry and James doesn't write poetry.
2. Either Emily writes poetry and James doesn't, or James writes poetry and Emily doesn't.
3. Dan and Mary are both correct.
4. Dan and Mary are both not correct.
5. Dan and Mary are not both correct.
6. Neither Dan is correct nor Mary is correct.
7. Either Dan and Mary are both correct or neither of them is correct.

Solution. First we identify and symbolize the propositional components occurring in each English statement. Then we will express the English statement in logical form.

1. *Emily writes poetry and James doesn't write poetry.*

   Let E represent "Emily writes poetry." The statement "James doesn't write poetry" is a short form for the sentence "James does not write poetry." Let J represent "James does write poetry." Then the logical form of the English statement is $E \wedge \neg J$.

2. *Either Emily writes poetry and James doesn't, or James writes poetry and Emily doesn't.*

   The word 'either' can be thought of as a warning that an 'or' is coming. The expression 'either X or Y' means that X is true or Y is true. So, the given English sentence can be expressed as

   (Emily writes poetry and James doesn't) or
   (James writes poetry and Emily doesn't).

   Using the propositions E and J given in our solution to item 1, the logical form of the English sentence can be expressed as $(E \wedge \neg J) \vee (J \wedge \neg E)$.

3. *Dan and Mary are both correct.*

   Let D represent "Dan is correct" and let M represent "Mary is correct." Thus, the logical form of the given statement is $D \wedge M$.

4. *Dan and Mary are both not correct.*

   In this sentence, the expression "both not" means that Dan is not correct and Mary is not correct. Let D and M be the propositions used in our solution to item 3. We conclude that the logical form of the sentence is $\neg D \wedge \neg M$.

5. *Dan and Mary are not both correct.*

   The expression "not both," in the above sentence, means that it is not the case that Dan and Mary are both correct. Let D and M be the propositions used in our solution to item 3. Consequently, the logical form of the sentence is $\neg(D \wedge M)$.

6. *Neither Dan is correct nor Mary is correct.*

   The word 'neither' can be thought of as a warning that a 'nor' is coming. The expression 'neither X nor Y' means that X is false and Y is also false. Let D and M be the propositions used in our solution to item 3. Then the logical form of the English sentence in item 6 is $\neg D \wedge \neg M$.

7. *Either Dan and Mary are both correct or neither of them is correct.*

   Since this English sentence begins with an 'either', it can be expressed as

   (Dan and Mary are both correct) or (neither of them is correct).

   Let D and M be as in our solution to item 3. The logical form of the English sentence in item 7 is given by $(D \wedge M) \vee (\neg D \wedge \neg M)$.

This completes our solution. Ⓢ

As stated in the preface, we use the symbol Ⓢ to mark the end of a solution.

Example 2. Analyze the logical forms of the following mathematical statements using only the mathematical relations "$<$" and "$=$" (look out for any possible hidden logical connectives).

1. $x \leq 4$.
2. $\sqrt{3} \not< 4$.
3. $1 \leq x \leq 4$.

Solution.

1. The statement $x \leq 4$ means that x is less than or equal to 4. So there is a hidden 'or' in this statement. The logical form of $x \leq 4$ can be expressed by $x < 4 \vee x = 4$.
2. The statement $\sqrt{3} \not< 4$ means that $\sqrt{3}$ is not less than 4. Therefore, the logical form of $\sqrt{3} \not< 4$ is $\neg(\sqrt{3} < 4)$.
3. Finally, the statement $1 \leq x \leq 4$ means that $1 \leq x$ and $x \leq 4$. So this statement contains the hidden connective 'and.' The assertion $1 \leq x \leq 4$ has the logical form $(1 < x \vee 1 = x) \wedge (x < 4 \vee x = 4)$. Ⓢ

1.1.2 Truth Tables

Given a collection of propositional components, say P, Q, and R, we can assign truth values to these components. For example, we can assign the truth values of P, Q, R to be T, F, T respectively, where T means "true" while F means "false." The truth value of a sentence in propositional logic can be evaluated from the truth values assigned to its components. We shall explain what this "means" by using truth tables. The logical connectives $\wedge$, $\vee$, $\neg$ yield the natural truth values given by the following three truth tables, respectively, in Table 1.1.

Truth table (1) has four rows (not including the header). The columns beneath P and Q list all the possible pairs of truth values that can be assigned to the components P and Q. For each such pair, the corresponding truth value for $P \wedge Q$ appears to the right. For example, consider the second pair of truth values in this table, T F. Thus, when the propositional components P and Q are assigned the respective truth values T and F, we see that the truth value of $P \wedge Q$ is F.

Truth table (2) asserts that when P and Q are assigned the respective truth values F and T, then the truth value of $P \vee Q$ is T. Furthermore, when P and Q are assigned

Table 1.1 Basic truth tables

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

(1) Conjunction

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

(2) Disjunction

P	$\neg P$
T	F
F	T

(3) Negation

the truth values T and T, then the truth value of $P \vee Q$ is also T. In mathematics, the connective "or" has the same meaning as "and/or," that is, $P \vee Q$ is true if either P is true *or* Q is true, or both P *and* Q are true. Thus, the assertion "$x \geq 2$ or $x \leq 2$" is true when $x = 1$, $x = 2$, or when $x = 3$. Our truth table for $P \vee Q$ reflects the fact that we are working in mathematics. (The word "or" in every day English sometimes excludes the possibility that P and Q could both be true.) Finally, the truth table (3) shows that the negation of a statement reverses the truth value of the statement.

Now that we know how to build truth tables for the sentences $P \wedge Q$, $P \vee Q$, and $\neg P$, we will discuss how to build truth tables for more complicated propositional sentences such as $(P \vee R) \wedge (\neg Q \wedge S)$.

Constructing Truth Tables for More Complicated Sentences

In Table 1.1, truth table (2) allows us to determine the truth value of $P \vee Q$ whenever we know the truth value of P and Q. In other words, the truth value of $P \vee Q$ is presented as a function of the truth values assigned to the components P and Q. In this section, we illustrate a method that will allow us to construct a truth table for any propositional sentence.

Given a propositional sentence one can identify the "outside" connective, that is, the "last connective one needs to evaluate." When the outside connective in a propositional sentence has been identified, one can then break up the sentence into its "parts." For example, in the propositional sentence $\neg P \vee (Q \wedge P)$ the logical connective $\vee$ is the outside connective with parts $\neg P$ and $Q \wedge P$. For another example, consider the propositional sentence $\neg(P \vee (Q \wedge P))$. We see that $\neg$ is the outside connective with corresponding part $P \vee (Q \wedge P)$.

Example 3. Construct a truth table that can be used to evaluate the truth value of the sentence $\neg P \vee (Q \wedge P)$ as a function of the truth values assigned to its components P and Q.

Solution. Of course, the components P and Q will each need a column in our truth table. Since there are two components, there will be four possible combinations of truth values for P and Q. We will enter these combinations in the two left-most columns in the same order as that in Table 1.1(1). The outside connective of the propositional sentence $\neg P \vee (Q \wedge P)$ is $\vee$. We can break this sentence into two parts $\neg P$ and $Q \wedge P$. So these parts will also need a column in our truth table. Since we can only break the sentences $\neg P$ and $Q \wedge P$ into components (namely, P and Q), we obtain the following truth table:

	P	Q	$\neg P$	$Q \wedge P$	$\neg P \vee (Q \wedge P)$
	T	T	F	T	T
	T	F	F	F	F
	F	T	T	F	T
	F	F	T	F	T
STEP #	1	1	2	3	4

We now describe in steps how we obtained the truth values in the above table. STEP 1: Specify all of the possible truth values that can be assigned to the components (resulting in four rows of truth values). STEP 2: In each row, use the truth value assigned to the component P to obtain the corresponding truth value for $\neg P$ by applying Table 1.1(3). STEP 3: In each row, use the truth values assigned to Q and P, to determine the corresponding truth value in the column under $Q \wedge P$, using Table 1.1(1). STEP 4: In each row, use the truth values in the columns under $\neg P$ and $Q \wedge P$ to obtain the matching truth value for the final column under the sentence $\neg P \vee (Q \wedge P)$ by employing Table 1.1(2). Ⓢ

In the construction of the above truth table, observe that whenever we broke up a sentence into parts, the columns for the parts appear to the left of the column for the sentence. We will do this again in our next example.

Example 4. Construct a truth table that can be used to evaluate the truth value of the sentence $P \wedge (Q \vee \neg R)$ as function of the truth values of the components P, Q, R.

Solution. We know that the components P, Q and R will each need a column in our truth table. Since there are three components, there will be eight possible truth value combinations for P, Q and R. The outside connective of the propositional sentence $P \wedge (Q \vee \neg R)$ is $\wedge$. We can break this sentence into two parts P and $Q \vee \neg R$. Since $Q \vee \neg R$ is not a component, it will need a column in our truth table. We now break up $Q \vee \neg R$ into the parts Q and $\neg R$. Because $\neg R$ is not a component, it will also require a column in our truth table. Thus, our desired truth table for $P \wedge (Q \vee \neg R)$ is

	P	Q	R	$\neg R$	$Q \vee \neg R$	$P \wedge (Q \vee \neg R)$
	T	T	T	F	T	T
	T	T	F	T	T	T
	T	F	T	F	F	F
	T	F	F	T	T	T
	F	T	T	F	T	F
	F	T	F	T	T	F
	F	F	T	F	F	F
	F	F	F	T	T	F
STEP #	1	1	1	2	3	4

We will now identify the steps that we used to obtain the truth values in the above table. STEP 1: Specify all of the possible truth values that can be assigned to the components (resulting in eight rows of truth values). STEP 2: In each row, use the truth value assigned to the component R to obtain the corresponding truth value for $\neg R$ by applying Table 1.1(3). STEP 3: In each row, use the truth values assigned to Q and $\neg R$, to determine the corresponding truth value in the column under $Q \vee \neg R$, using Table 1.1(2). STEP 4: In each row, use the truth values in the columns under P and $Q \vee \neg R$ to obtain the matching truth value for the final column under the sentence $P \wedge (Q \vee \neg R)$. Ⓢ

1.1.3 Tautologies and Contradictions

Suppose, after constructing a truth table for a propositional sentence, you see that each entry in the final column is true. This indicates a situation where the sentence is true no matter what truth values are assigned to its components. When this occurs, the sentence is called a tautology.

Definition 1.1.1. We shall say that a propositional sentence is a **tautology** when its truth value is true regardless of the truth values of its components.

Thus, a propositional sentence is a tautology if it is always true. For example, one can see from the following truth table that the sentence $P \vee \neg P$ is a tautology.

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

Definition 1.1.2. We shall say that a propositional sentence is a **contradiction** when its truth value is false regardless of the truth values of its components.

In other words, a propositional sentence is a contradiction if it is always false. One can easily show that the sentence $P \wedge \neg P$ is a contradiction.

1.1.4 Logical Equivalence

The following definition describes when two propositional sentences are logically equivalent, that is, when they mean the same thing. Mathematicians frequently take advantage of logical equivalence to simplify their proofs and we shall do the same in this book. We will use Greek letters (e.g., α, β, φ and ψ – see page xv) to represent propositional sentences.

Definition 1.1.3. Let ψ and φ be two sentences of propositional logic. We say that ψ and φ are **logically equivalent**, denoted by $\psi \Leftrightarrow \varphi$, when the following holds:

For every truth assignment applied to the components of ψ and φ, the resulting truth values of ψ and φ are identical.

Let φ and ψ be propositional sentences with the same components. Construct truth tables for φ and ψ so that each component has the same column in both tables. Suppose that these two truth tables also have the same final column. We can then conclude that φ and ψ are logically equivalent. Thus, φ and ψ are "both true at the same time and both false at the same time."

Example 5. Let ψ be the sentence $\neg(P \vee Q)$ and let φ be the sentence $\neg P \wedge \neg Q$. Show that ψ and φ are logically equivalent, that is, show $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$.

Solution. After constructing individual truth tables for the statements $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$, we obtain

P	Q	$P \vee Q$	$\neg(P \vee Q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

P	Q	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

So each truth assignment applied to the components P and Q yields the same truth value for $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$. Therefore, we have that $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$. In other words, since the final columns of the truth tables for $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$ are the identical, we can conclude that they are logically equivalent. Ⓢ

When φ and ψ are logically equivalent, we will say that $\psi \Leftrightarrow \varphi$ is a *logic law*. We now present two important logic laws that are often used in mathematical proofs. These laws were first identified by Augustus De Morgan (see Example 5).

De Morgan's Laws (DML)

1. $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$.
2. $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$.

Let ψ and φ be two sentences of propositional logic. If one can apply a truth assignment to the components of ψ and φ so that the resulting truth values of ψ and φ disagree, then ψ and φ are *not logically equivalent*. We will use this fact in our next example which shows that the placement of parentheses in a propositional sentence is very important. A regrouping can change the meaning of the sentence.

Example 6. Show that $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee R$ are not logically equivalent.

Solution. We shall use the truth table

P	Q	R	$P \wedge (Q \vee R)$	$(P \wedge Q) \vee R$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

Since their final columns are not the identical, we see that $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee R$ are not equivalent. In particular, the truth assignment to the components in row 5 yields different truth values for $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee R$. Ⓢ

1.1.5 Propositional Logic Laws

Propositional logic will be used as a tool to help us develop both the structure and the presentation of mathematical proofs. Listed below are the important laws of logic that will allow us to simplify more complicated propositional sentences and to streamline the presentation of some mathematical proofs. In Section 1.1.6, we will also use these logic laws to derive new logic laws without the use of truth tables.

De Morgan's Laws (DML)

1. $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$.
2. $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$.

Commutative Laws

1. $P \wedge Q \Leftrightarrow Q \wedge P$.
2. $P \vee Q \Leftrightarrow Q \vee P$.

Associative Laws

1. $P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$.
2. $P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$.

Idempotent Laws

1. $P \wedge P \Leftrightarrow P$.
2. $P \vee P \Leftrightarrow P$.

Distributive Laws

1. $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$.
2. $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$.

3. $(Q \vee R) \wedge P \Leftrightarrow (Q \wedge P) \vee (R \wedge P)$.
4. $(Q \wedge R) \vee P \Leftrightarrow (Q \vee P) \wedge (R \vee P)$.

Double Negation Law (DNL)

1. $\neg\neg P \Leftrightarrow P$.

Tautology Law

1. $P \wedge (\text{a tautology}) \Leftrightarrow P$.

Contradiction Law

1. $P \vee (\text{a contradiction}) \Leftrightarrow P$.

We now give examples of the above Tautology Law and Contradiction Law. First recall that $Q \vee \neg Q$ is a tautology. From the Tautology Law we obtain the following logical equivalence

$$P \wedge (Q \vee \neg Q) \Leftrightarrow P.$$

On the other hand, because $Q \wedge \neg Q$ is a contradiction, we conclude that

$$P \vee (Q \wedge \neg Q) \Leftrightarrow P$$

by the Contradiction Law.

1.1.6 Logic Laws and Substitution

Consider the algebraic identity $(x-y)(x+y) = x^2 - y^2$. If we replace x with ab, then we obtain another algebraic identity $(ab-y)(ab+y) = (ab)^2 - y^2$. Similarly, if a propositional component appears in a logic law and we replace all occurrences of this component with a propositional sentence, then we will obtain another logic law. For example, consider the distributive law

$$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R).$$

Let us replace P with $\neg A$ and replace Q with a propositional sentence ψ. We then obtain the following logical equivalence, which is an application of the Distributive Law:

$$\neg A \vee (\psi \wedge R) \Leftrightarrow (\neg A \vee \psi) \wedge (\neg A \vee R).$$

In addition, let α and β be propositional sentences that are logically equivalent, that is, $\alpha \Leftrightarrow \beta$. If α appears in a given propositional sentence Θ and we replace occurrences of α in Θ with β, then the resulting new sentence will be logically equivalent to Θ. For example, consider the sentence Θ given by $\neg Q \vee \alpha$ and suppose that $\alpha \Leftrightarrow \beta$. Upon replacing α with β, we obtain the new sentence $\neg Q \vee \beta$. We can conclude that $(\neg Q \vee \alpha) \Leftrightarrow (\neg Q \vee \beta)$, because $\alpha \Leftrightarrow \beta$.

Example 7. Using logic laws, find a simpler sentence equivalent to the formula $\neg Q \vee \neg(\neg P \vee \neg Q)$.

Solution. We start with $\neg Q \vee \neg(\neg P \vee \neg Q)$ and apply logic laws as follows:

$$\begin{aligned}
\neg Q \vee \neg(\neg P \vee \neg Q) &\Leftrightarrow \neg Q \vee (\neg\neg P \wedge \neg\neg Q) && \text{by De Morgan's Law} \\
&\Leftrightarrow \neg Q \vee (P \wedge Q) && \text{by Double Negation Law} \\
&\Leftrightarrow (\neg Q \vee P) \wedge (\neg Q \vee Q) && \text{by Distributive Law} \\
&\Leftrightarrow (\neg Q \vee P) && \text{by Tautology Law.}
\end{aligned}$$

Therefore, $\neg Q \vee \neg(\neg P \vee \neg Q) \Leftrightarrow \neg Q \vee P$. Thus, $\neg Q \vee P$ is a simplified version of $\neg Q \vee \neg(\neg P \vee \neg Q)$. Ⓢ

Example 8. Using propositional logic laws, show that $\neg(P \wedge \neg Q) \Leftrightarrow (\neg P \vee Q)$.

Solution. We start with the more complicated side $\neg(P \wedge \neg Q)$ and derive the simpler side as follows:

$$\begin{aligned}
\neg(P \wedge \neg Q) &\Leftrightarrow (\neg P \vee \neg\neg Q) && \text{by De Morgan's Law} \\
&\Leftrightarrow (\neg P \vee Q) && \text{by Double Negation Law.}
\end{aligned}$$

Therefore, $\neg(P \wedge \neg Q) \Leftrightarrow (\neg P \vee Q)$. Ⓢ

Exercises 1.1

1. Only one of the following is a tautology. Which one is it?

(a) $(P \vee \neg P) \wedge Q$.
(b) $(P \vee \neg P) \vee Q$.

2. Using one of De Morgan's Laws, write a negation of the statement: *Ron runs on Thursdays and Pete plays poker on Saturdays.* Express you answer in English.

3. Use one of De Morgan's Laws to write a negation, in English, of the statement: *My computer program has an error or the wrong value is assigned to a constant.*

4. Using propositional logic laws (see Section 1.1.5), supply a law justifying each step:

$$\begin{aligned}
(P \vee \neg Q) \wedge (\neg P \vee \neg Q) &\Leftrightarrow (\neg Q \vee P) \wedge (\neg Q \vee \neg P) && \text{by } \underline{\qquad\qquad} \\
&\Leftrightarrow \neg Q \vee (\neg P \wedge P) && \text{by } \underline{\qquad\qquad} \\
&\Leftrightarrow \neg Q && \text{by } \underline{\qquad\qquad}
\end{aligned}$$

5. Using the propositional logic laws in Section 1.1.5, find simpler sentences (see Example 7) that are equivalent to the following:

(a) $\neg(\neg P \wedge \neg Q)$.
(b) $\neg Q \wedge \neg(\neg P \wedge \neg Q)$.
(c) $\neg(\neg P \vee Q) \vee (P \wedge \neg R)$.

6. Which of the following statements are true and which are false?

(a) $(\pi^2 > 9) \wedge (\pi > 3)$.
(b) $(\pi^2 > 9) \vee (\pi > 3)$.
(c) $(\sin(2\pi) > 9) \vee (\sin(2\pi) < 0)$.
(d) $(\sin(\pi) > 9) \vee \neg(\sin(\pi) \leq 0)$.

7. Using truth tables, show that $(\neg P \vee Q) \vee (P \wedge \neg Q)$ is a tautology. What can you conclude about the sentence $\neg((\neg P \vee Q) \vee (P \wedge \neg Q))$?

8. Using propositional logic laws, show that $P \vee (Q \wedge \neg P) \Leftrightarrow P \vee Q$.

9. Using logic laws, show that $\neg(P \vee \neg Q) \vee (\neg P \wedge \neg Q) \Leftrightarrow \neg P$.

1.2 The Conditional and Biconditional Connectives

1.2.1 Conditional Statements

Many mathematical theorems have the form "if P, then Q" or, equivalently, "P implies Q." Here is one important example that you may have seen in your calculus course:

Theorem. *If f is differentiable at the point a, then f is continuous at a.*

Let D be the proposition "f is differentiable at the point a" and let C be the proposition "f is continuous at a." The theorem can now be expressed as

Theorem. *If D, then C.*

A *conditional statement* has the form "if P, then Q." The statement P is called the **hypothesis** and the statement Q is called the **conclusion**. Thus, a conditional statement asserts that the truth of the hypothesis "implies" the truth of the conclusion. This is such an important idea in mathematics that we will now introduce a logical connective which will capture the mathematical notion that the hypothesis implies the conclusion.

The Conditional Connective. Given propositions P and Q, the conditional connective $\to$ means "implies" and can be used to form the sentence $P \to Q$. The sentence $P \to Q$ can be read as "P implies Q" or "if P, then Q."