Preface

As an undergraduate, one of my most memorable and thrilling moments occurred when I first discovered how to prove an assigned theorem in one of my upper level mathematics courses. I also noticed that a few of my fellow students were having trouble proving this theorem. Even today, after being successful in calculus, many students have difficulty with proofs in their upper-division mathematics courses. To be successful in these more advanced courses, students must possess three essential skills: the ability to read, to understand, and to communicate in the language of mathematics. I wrote this book specifically to help students acquire these important skills and to enhance their ability to formulate and construct mathematical proofs.

When I was in college, what was it that helped me to discover and write mathematical proofs? Before enrolling in my first upper division mathematics course, I completed a beginning course in logic offered by the philosophy department. This course introduced me to formal proofs in a natural deduction system. During my first upper division mathematics course, I soon realized that I could apply the ideas that I learned in the logic course to help me write and find mathematical proofs. This book is intended to show students how basic logical principles can also help them to discover and compose mathematical proofs.

The core topics covered in the text are logic, sets, relations, functions and induction. Logic is covered, not as an end in itself, but as an instrument for analyzing the logical structure of mathematical assertions and as a tool for constructing valid mathematical proofs. I do not presume that the reader has an intuitive understanding of the principles of reasoning that many mathematicians take for granted.

Every theorem in mathematics must have a proof from a set of stated assumptions. The proof demonstrates that the conclusion of the theorem follows from the assumptions by the laws of logic alone. Can the notions of “laws of logic” and “proof” be made precise? This text is devoted to establishing a positive answer to this question. Students are given a plan of attack for finding and composing a correct proof of a given mathematical theorem. This is done by developing a method for generating "proof diagrams."[1] For example, a proof of the statement $(\forall n \geq 1)\, P(n)$ by induction, uses the logical structure illustrated in the proof diagram

Prove $P(1)$.
Let $n \geq 1$ be an integer.
Assume $P(n)$.
Prove $P(n+1)$.

where indentation is used to display a proof's logical dependencies.

Diagramming a proof is a way of presenting the relationships between the various parts of a proof. The resulting proof diagram clearly demonstrates the structure of the proof and provides a tool for showing students how to write a correct mathematical proof. Analyzing a proof and portraying its *logical* structure with a consistent visual scheme can be helpful – both for proof beginners and for those trying to make sense of a proof at any level. Many students feel more confident using this well-structured approach for finding and writing proofs.

Each student of mathematics needs to learn how to find and write mathematical proofs. These are probably two of the most difficult skills that a mathematics major has to develop. Students often fail to construct a proof of a mathematical statement because they lack confidence or just do not know how to get started. This text is designed to increase students confidence by showing them how to first break up the statement into its logical parts and then use these parts as a guide for finding and writing a proof. Even with a guide, the work required to find a proof can be quite challenging. Professional mathematicians also have difficulty finding proofs; however, mathematicians know that persistence often pays off and thus, they do easily not give up.

> Patience and perseverance have a magical effect before which difficulties disappear and obstacles vanish. – John Quincy Adams

> Keep on going, and the chances are that you will stumble on something, perhaps when you are least expecting it. – Charles Kettering

What Is Covered?

The book is intended for students who want to learn how to prove theorems and be better prepared for the rigors required in their future mathematics courses. The first two chapters introduce students to logical connectives, truth tables, inference rules, deductions, quantifiers, variables, and truth sets. Chapter 1 focuses on propositional logic and Chapter 2 presents the logic of quantifiers. This initial emphasis on logic

[1]School children were once taught to diagram an English sentence as a means to analyze its grammatical structure.

is motivated only by the desire to show the reader how to discover, develop, and compose logically correct proofs.

Chapter 3 methodically presents the key strategies that are used in mathematical proofs. Each strategy is presented as a proof diagram and specifically responds to the logical form that a given mathematical statement may have. Furthermore, each proof strategy is carefully illustrated by a variety of mathematical theorems concerning the natural, rational and real numbers. The remaining chapters of the book presume the material presented in Chapter 3.

The attention in Chapter 4 is on proof by mathematical induction and concludes with a proof of the fundamental theorem of arithmetic.

Chapters 5–7 will introduce students to the essential concepts that appear in all branches of mathematics. Chapter 5 covers set theory and proofs about sets. In particular, I present the proof strategies that are used to prove set equality and to prove the subset relation. Functions are the main topic in Chapter 6, with a concentration on strategies for proving when functions are well-defined, one-to-one, and onto. With this foundation, Chapter 6 ends on the topic of countable and uncountable sets. Many of our proofs, on countability, presume the fundamental theorem of arithmetic. Chapter 7 presents equivalence relations, partitions, congruence modulo m relations, modular arithmetic, and partially ordered sets.

In the final two chapters, the objective is to better prepare students for abstract algebra and real analysis. Students will be introduced to some of the key topics that will be covered in their real analysis and abstract algebra courses; moreover, I do not just offer a preview of the basic concepts that will be addressed in these courses. The main goal is to give students some fundamental tools that will increase their likelihood of success.

A student's first course with a heavy emphasizes on proof is usually abstract algebra. The main aim of Chapter 8 is to prepare students for some of the important topics that will be covered in such a course. I first discuss algebraic structures and then move on to groups, subgroups, and normal subgroups. I provide proof strategies for dealing with these latter two concepts, as well. Using the results of Chapter 6 on one-to-one and onto functions, I also investigate permutation groups and the symmetric group. Chapter 8 also introduces rings and then ends on the topics of quotient algebras, quotient groups, and quotient rings. These final topics presume the material on equivalence relations and partitions covered in Chapter 7.

In real analysis, a facility for working with the supremum of a bounded set, the limit of a sequence, and the ε-δ definitions of continuity is essential for a student to be successful. Many students stumble when first asked to compose proofs using these core definitions. Chapter 9 is designed to better prepare students and allow them to overcome these initial hurdles. I present proof strategies that explicitly show students how to deal with the fundamental definitions that they will encounter in real analysis; followed by numerous examples of proofs that use these strategies.

Exercises are given at the end of each section in a chapter. Suggestions are also provided for those exercises that a newcomer to mathematical proof may find more challenging. The symbol Ⓢ marks the end of a solution, the symbol Ⓐ indicates the end of a proof analysis, and the symbol □ is used to identify the end of a proof.

Using This Text in a Transition Course

A standard transition course is designed to better prepare students for real analysis and abstract algebra. Such a course should cover Chapters 1–7. The basics of logic and proof are covered in Chapters 1–3. These first three chapters offer a basis for all of the material covered in the text. My experience has been that students easily grasp the topics covered in Chapters 1 and 2. Consequently, these first two chapters can usually be covered at a quick pace.

In Chapter 4 one could cover Sections 4.1–4.6 and then simply state the fundamental theorem of arithmetic, which is proven in Section 4.7. Furthermore, if students are already well versed in summation and factorial notation, Section 4.3 may be skipped. The topics covered in Chapters 5–6 are essential for any student of mathematics; however, Sections 5.4 (the axioms of set theory) and 6.5 (countable and uncountable sets) can be omitted as they are not used anywhere else in the book. Chapter 7 introduces equivalence relations, partitions, and congruence relations on the integers. Since the material in this chapter may be new to many students, these important topics should not be overlooked.

Chapters 8 and 9 are independent of each other. In Chapter 8, if time is limited, one could first introduce students to the notion of an algebraic structure and then focus their attention on groups by covering only Section 8.3, where proof strategies are presented that deal with subgroups and normal subgroups. Alternatively, after discussing algebraic structures, one could just introduce the ring concept by covering Section 8.5, as this section does not presume the group concept. Chapter 9 presents: (1) the supremum and infimum of a bounded set of real numbers, (2) the limit of a sequence, and (3) continuity of a function. Since these three topics are developed independently, one could discuss any combination of these concepts. In any case, it is hoped that students will view these final two chapters, and the entire book, as a useful resource in their future courses.