In other words, use the diagram

Let $x, y, z \in A$.
  Assume $x \sim y$.
  Assume $y \sim z$.
    Prove $x \sim z$.

Example 8. Define the relation $\sim$ on the set of natural numbers $\mathbb{N}$ by

$$x \sim y \text{ if and only if } x = yk \text{ for some } k \in \mathbb{N}. \tag{7.1}$$

Determine whether or not the relation $\sim$ is reflexive, symmetric, or transitive. You must justify your answers with a proof if the relation is reflexive, symmetric, or transitive. You must provide a counterexample if the relation fails to be reflexive, symmetric, or transitive.

Solution. We have the relation $\sim$ on the set $\mathbb{N}$ defined by (7.1).

- The relation $\sim$ is reflexive.
  Proof. Let x be a natural number. Since $x = x \cdot 1$ where $1 \in \mathbb{N}$, we conclude that $x \sim x$. □
- The relation $\sim$ is not symmetric. Let $x = 6$ and $y = 2$. Since $x = y \cdot 3$, we see that $x \sim y$. We also see that $y \not\sim x$ because $2 \neq 6 \cdot k$ for any $k \in \mathbb{N}$.
- The relation $\sim$ is transitive.

  Proof. Let x, y, z be natural numbers. Assume $x \sim y$ and $y \sim z$. Thus, (1) $x = yk$ and (2) $y = zj$ for some $k, j \in \mathbb{N}$. By substituting the value for y in equation (2) into equation (1), we obtain $x = (zj)k = z(jk)$. Therefore, $x = z(jk)$ where $jk \in \mathbb{N}$. Hence, $x \sim z$. □

This completes our solution. Ⓢ

Exercises 7.1

1. Let $\sim$ be the relation on the set $A = \{0, 1, 2, 3, 4, 5\}$ defined by $a \sim b$ if and only if $a \mid (b+1)$. Represent this relation as a set of ordered pairs.
2. Let $\sim$ be the relation on the set $\mathbb{R}$ defined by $x \sim y$ if and only if $x + y \geq 0$. Prove that this relation is symmetric. Find counterexamples showing that this relation is not reflexive and not transitive.
3. Let $\sim$ be the relation on the set $\mathbb{R}$ defined by $x \sim y$ if and only if $xy \geq 0$. Prove that this relation is reflexive and symmetric. Find a counterexample showing that this relation is not transitive.
4. The relation $\sim$ on the set $\mathbb{Z}$ is defined by $m \sim n$ if and only if $m - n$ is even. Prove that this relation is reflexive, symmetric, and transitive.

5. The relation $\sim$ on the set $\mathbb{Z}$ is defined by $m \sim n$ if and only if $m - n$ is odd. Prove that this relation is symmetric. Find counterexamples showing that this relation is not reflexive and not transitive.

6. The relation $\sim$ on the set $\mathbb{R}$ is defined by $x \sim y$ if and only if $|x| = |y|$. Prove that this relation is reflexive, symmetric, and transitive.

7. The relation $\sim$ on the set $\mathbb{Z}$ is defined by $m \sim n$ if and only if $3 \mid (m - n)$. Prove that this relation is reflexive, symmetric, and transitive.

8. Let the relation $\sim$ on the set $\mathbb{N}$ be defined by $m \sim n$ if and only if $m \mid n$. Prove that this relation is reflexive and transitive. Find a counterexample showing that this relation is not symmetric.

9. The relation $\sim$ on the set $\mathbb{R}$ is defined by $x \sim y$ if and only if $\sin(x) = \sin(y)$. Prove that this relation is reflexive, symmetric, and transitive.

7.2 Equivalence Relations and Partitions

Because the equality relation has been so useful, mathematicians have generalized this concept. A relation is called an equivalence relation if it satisfies the three key properties that are normally associated with equality.

Definition 7.2.1. A relation $\sim$ on a set A is called an **equivalence relation** if it is reflexive, symmetric, and transitive.

Thus, to prove that a relation is an equivalence relation, three distinct proofs are required; that is, one must prove that the relation is (1) reflexive, (2) symmetric, and (3) transitive. Equivalence relations are used in many areas of mathematics. An equivalence relation allows one to connect those elements of a set that have a particular property in common. Example 1, below, identifies three equivalence relations. The relation in item 1 joins the even integers and links the odd integers, the relation in item 2 associates the real numbers that have the same absolute value, and the equivalence relation in item 3 unites those integers (as we will see) that have the same remainder when divided by 3.

Example 1. One can show that each one of the relations below is an equivalence relation (see Exercises 4, 6 and 7 of Section 7.1):

1. The relation $\sim$ on the set $\mathbb{Z}$ defined by $m \sim n$ if and only if $m - n$ is even.
2. The relation $\sim$ on the set $\mathbb{R}$ defined by $x \sim y$ if and only if $|x| = |y|$.
3. The relation $\sim$ on the set $\mathbb{Z}$ defined by $m \sim n$ if and only if $3 \mid (m - n)$.

The main result that we will establish in the section is that an equivalence relation on a set A induces a partition of A into disjoint subsets. This will allow us to create a new mathematical object from an old one. For each $a \in A$, we must first form the set of all those elements in A that are related to a.