Therefore, $f^{-1}[U] \cup f^{-1}[V] \subseteq f^{-1}[U \cup V]$. This completes the proof of (d). □

Theorem 6.4.6. *Let $f\colon X \to Y$ be a function. Let C, D be subsets of X. If f is one-to-one, then $f[C \cap D] = f[C] \cap f[D]$.*

Proof. Let $f\colon X \to Y$ be a function. Let C, D be subsets of X and assume that f is one-to-one. We shall prove that $f[C \cap D] = f[C] \cap f[D]$. Theorem 6.4.5(a) implies that $f[C \cap D] \subseteq f[C] \cap f[D]$. To show that $f[C] \cap f[D] \subseteq f[C \cap D]$, let $y \in f[C] \cap f[D]$. We will prove that $y \in f[C \cap D]$. Since $y \in f[C] \cap f[D]$, we see that $y \in f[C]$ and $y \in f[D]$. Because $y \in f[C]$, there is a $c \in C$ such that $f(c) = y$. Also, since $y \in f[D]$, there is a $d \in D$ such that $f(d) = y$. Hence, $y = f(c) = f(d)$. Since f is one-to-one, we have $c = d$. Thus, $c \in D$. So $c \in C \cap D$ and therefore, $y = f(c) \in f[C \cap D]$. We conclude that $f[C \cap D] = f[C] \cap f[D]$. □

Exercises 6.4

1. Using Definitions 6.4.1 and 6.4.2, explain why items 1–4 of Remark 6.4.3 hold.
2. Prove Theorem 6.4.4.
3. Prove item (b) of Theorem 6.4.5.
4. Prove item (c) of Theorem 6.4.5.
5. Given $a, b \in \mathbb{R}$ with $a > 0$, define the function $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = ax + b$. Let $U = [2, 3]$. Using interval notation, evaluate $f[U]$ and $f^{-1}[U]$.
6. Define the function $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ and let $U = [-1, 4]$. Show the following:
   (a) $f[f^{-1}[U]] \neq U$.
   (b) $f^{-1}[f[U]] \neq U$.
   (c) $f[f^{-1}[U]] \neq f^{-1}[f[U]]$.
7. Let $f\colon X \to Y$ be a function and let $A \subseteq X$ and $B \subseteq X$. Prove that if $A \subseteq B$, then $f[A] \subseteq f[B]$.
8. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined in Example 1 on page 189. Find $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ such that $f[A] \subseteq f[B]$ and $A \not\subseteq B$.
9. Suppose $f\colon X \to Y$ is a one-to-one function. Let $A \subseteq X$ and $B \subseteq X$. Prove that if $f[A] \subseteq f[B]$, then $A \subseteq B$.
10. Let $f\colon X \to Y$ be a function and let $C \subseteq Y$ and $D \subseteq Y$. Prove that if $C \subseteq D$, then $f^{-1}[C] \subseteq f^{-1}[D]$.
11. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined in Example 3. Find $C \subseteq \mathbb{R}$ and $D \subseteq \mathbb{R}$ such that $f^{-1}[C] \subseteq f^{-1}[D]$ and $C \not\subseteq D$.
12. Suppose $f\colon X \to Y$ is onto and let $C \subseteq Y$ and $D \subseteq Y$. Prove if $f^{-1}[C] \subseteq f^{-1}[D]$, then $C \subseteq D$.
13. Let $f\colon X \to Y$ be a function. Let A be a subset of X. Prove that $A \subseteq f^{-1}[f[A]]$.