$$\begin{aligned}(f \circ g)(x) &= f(g(x)) && \text{by definition of composition}\\ &= f(y) && \text{because } g(x) = y\\ &= z && \text{because } f(y) = z. \qquad \square\end{aligned}$$

Exercises 6.3

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 - 1$ and let $g\colon \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^3 + 1$. Evaluate the values:

(a) $(f \circ g)(1)$
(b) $(g \circ f)(1)$
(c) $(f \circ f)(1)$
(d) $(g \circ g)(1)$.

2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 - 1$ and let $g\colon \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^3 + 1$. Obtain formulas for the following compositions:

(a) $(f \circ g)(x)$
(b) $(g \circ f)(x)$
(c) $(f \circ f)(x)$
(d) $(g \circ g)(x)$.

3. Let $f\colon A \to B$ and $g\colon B \to A$ be functions. Suppose that $f(g(b)) = b$ for all $b \in B$ and $g(f(a)) = a$ for all $a \in A$. Prove that f and g are one-to-one and onto.

4. For $a, b \in \mathbb{R}$ with $a \neq 0$, define the function $T_{a,b}\colon \mathbb{R} \to \mathbb{R}$ by $T_{a,b}(x) = ax + b$. Let G be the set of all such functions, that is, let $G = \{T_{a,b} : a, b \in \mathbb{R} \text{ and } a \neq 0\}$.

(a) Let $T_{a,b} \in G$ and $T_{c,d} \in G$. Show that $T_{a,b} \circ T_{c,d} = T_{ac,ad+b}$.
(b) Let $T_{a,b} \in G$ and $T_{c,d} \in G$. Show that $(T_{a,b} \circ T_{c,d}) \in G$.
(c) Let $T_{a,b} \in G$. Prove that $T_{a,b}$ is one-to-one and onto.
(d) Let $I\colon \mathbb{R} \to \mathbb{R}$ be the identity function. Show that $I \in G$.
(e) Let $T_{a,b} \in G$. Show that $T_{a,b}^{-1} = T_{\frac{1}{a}, -\frac{b}{a}}$ and thus, $T_{a,b}^{-1} \in G$.
(f) Find a $T_{a,b} \in G$ and $T_{c,d} \in G$ so that $(T_{a,b} \circ T_{c,d}) \neq (T_{c,d} \circ T_{a,b})$.

5. Given $a \in \mathbb{Q}$ with $a \neq 0$ and $b \in \mathbb{R}$, define $T_{a,b}\colon \mathbb{R} \to \mathbb{R}$ by $T_{a,b}(x) = ax + b$. Let $H = \{T_{a,b} : a \in \mathbb{Q},\ b \in \mathbb{R} \text{ and } a \neq 0\}$.

(a) For any $T_{a,b} \in H$ and $T_{c,d} \in H$, show that $(T_{a,b} \circ T_{c,d}) \in H$.
(b) Let $I\colon \mathbb{R} \to \mathbb{R}$ be the identity function. Show that $I \in H$.
(c) Let $T_{a,b} \in H$. Show that $T_{a,b}^{-1} \in H$.

6. Let $g\colon A \to B$ and $f\colon B \to C$. Suppose that $(f \circ g)\colon A \to C$ is one-to-one. Prove that g is one-to-one.

7. Let $g\colon A \to B$ and $f\colon B \to C$. Suppose that $(f \circ g)\colon A \to C$ is one-to-one and g is onto. Prove that f is one-to-one.

8. Let $f: B \to C$ and $g: A \to B$. Suppose that $(f \circ g): A \to C$ is onto. Prove that f is onto.

9. Let $g: A \to B$ and $f: B \to C$. Suppose that $(f \circ g): A \to C$ is onto and f is one-to-one. Prove that g is onto.

10. Let $h: A \to B$, $g: B \to C$ and $f: C \to D$. Prove that $(f \circ g) \circ h = f \circ (g \circ h)$.

Exercise Notes: For Exercise 4(a), find a formula for $T_{a,b}(T_{c,d}(x))$. For Exercise 4(e), find a formula for $T_{a,b}^{-1}$.

6.4 Functions Acting on Sets

There are times when we are more interested in what a function does to an entire subset of its domain, rather than how it affects an individual element in the domain. Understanding this behavior on sets can allow one to better understand the function itself and can reveal some properties concerning its domain and range. The concept of a function "acting on a set," is one that appears in every branch of mathematics.

Definition 6.4.1 (Image of a Set). Let $f: X \to Y$ be a function. Let $S \subseteq X$. The set $f[S]$, called the **image** of S, is defined by

$$f[S] = \{f(x) : x \in S\} = \{y \in Y : y = f(x) \text{ for some } x \in S\}.$$

Figure 6.9 illustrates Definition 6.4.1. The square S represents a subset of the domain of the function f. The image $f[S]$, represented by the rectangle, is the set of all values of the function that are obtained from the inputs that are in the set S.

Example 1. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|$ and $S = \{-12, -3, 2, 3\}$. Then the image of S is $f[S] = \{f(x) : x \in S\} = \{|x| : x \in S\} = \{2, 3, 12\}$. Observe that $f(12) \in f[S]$ and yet $12 \notin S$.

Example 2. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ and $S = \{-4, -3, 2, 3\}$. Then the image of S is $f[S] = \{f(x) : x \in S\} = \{x^2 : x \in S\} = \{16, 9, 4\}$. Let U be the interval $[-2, 3]$. Then $f[U] = \{f(x) : x \in U\} = \{x^2 : -2 \leq x \leq 3\} = [0, 9]$.

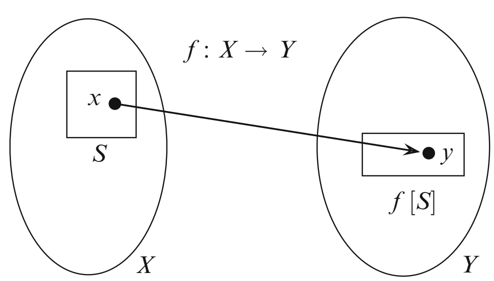

Fig. 6.9 Starting with $S \subseteq X$ we can construct the image $f[S] \subseteq Y$