Example 2. One can show that the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = (x-4)^3 + 2$ is one-to-one and onto. Find a formula for the inverse function f^{-1}.

Solution. After solving the equation $(x-4)^3 + 2 = y$ for x, we obtain $x = 4 + \sqrt[3]{y-2}$. Therefore, $f^{-1}(y) = 4 + \sqrt[3]{y-2}$ is the formula for the inverse function. One can now show that $f(x) = y$ if and only if $f^{-1}(y) = x$, for all $x, y \in \mathbb{R}$. Ⓢ

Unfortunately, the above procedure for finding a formula for an inverse function can fail. For example, consider the one-to-one and onto function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^5 + x^3$. It is impossible, using radicals, to algebraically solve the equation $x^5 + x^2 = y$ for x and thus, there is no elementary formula for f^{-1}.

Theorem 6.2.14. *Suppose $f\colon A \to B$ is one-to-one and onto. Let $f^{-1}\colon B \to A$ be the inverse of f. Then f^{-1} is also one-to-one and onto.*

Proof. Let $f\colon A \to B$ be one-to-one and onto. We first prove that $f^{-1}\colon B \to A$ is one-to-one. Let $b, b' \in B$. Assume $f^{-1}(b) = f^{-1}(b')$. Let $a \in A$ be this common value. Thus, $f^{-1}(b) = a$ and $f^{-1}(b') = a$. So $f(a) = b$ and $f(a) = b'$, by (6.12). Since f is a function, we conclude that $b = b'$. Hence, f^{-1} is one-to-one.

To prove that $f^{-1}\colon B \to A$ is onto, let $a \in A$. So there is a $b \in B$ be such that $f(a) = b$. By (6.12), $f^{-1}(b) = a$. Therefore, f^{-1} is onto. □

Exercises 6.2

1. Define $f\colon \mathbb{Z} \to \mathbb{Z}$ by $f(n) = 3n + 2$.
   (a) Is f one-to-one? Prove it, or provide a counterexample.
   (b) Is f onto? Prove it, or provide a counterexample.
2. Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.
   (a) Is f one-to-one? Prove it, or provide a counterexample.
   (b) Is f onto? Prove it, or provide a counterexample.
3. Define a function $f\colon \mathbb{N} \to \mathbb{N}$ that is one-to-one but not onto.
4. Define a function $f\colon \mathbb{N} \to \mathbb{N}$ that is onto but not one-to-one.
5. Let $A = \{x \in \mathbb{R} : x \neq -1\}$. Define $f\colon A \to \mathbb{R}$ by $f(x) = \frac{2x}{x+1}$. Prove that f is one-to-one.
6. Let $A = \{x \in \mathbb{R} : x \neq 1\}$. Define $f\colon A \to \mathbb{R}$ by $f(x) = \frac{3x}{2x-2}$. Prove f is not onto.
7. Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = x - x^3$. Is f one-to-one? Is it onto?
8. Let $A = \{x \in \mathbb{R} : x \neq 2\}$ and let $B = \{y \in \mathbb{R} : y \neq 4\}$. Define $f\colon A \to B$ by $f(x) = \frac{4x}{x-2}$. Prove that f is onto.
9. Let $A = \{x \in \mathbb{R} : x \neq 2\}$ and let $B = \{y \in \mathbb{R} : y \neq 4\}$. Prove the function $f\colon A \to B$ defined by $f(x) = \frac{4x}{x-2}$ is one-to-one.
10. Let $a, b \in \mathbb{R}$ with $a \neq 0$ and define the function $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = ax + b$. Given that f is one-to-one and onto, find a formula for $f^{-1}\colon \mathbb{R} \to \mathbb{R}$.

11. Prove that the function $f\colon \mathbb{R} \to \mathbb{R}$, defined below, is one-to-one and onto.

$$f(x) = \begin{cases} x^2, & \text{if } x \geq 0; \\ -x^2, & \text{if } x < 0. \end{cases} \tag{6.13}$$

12. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by (6.13) in Exercise 11. Given that f is one-to-one and onto, find a formula for the inverse function $f^{-1}\colon \mathbb{R} \to \mathbb{R}$.

13. Suppose that $f\colon \mathbb{R} \to \mathbb{R}^+$ is one-to-one. Define $g\colon \mathbb{R} \to \mathbb{R}^+$ by $g(x) = (f(x))^2$. Prove that g is one-to-one.

14. Suppose that $f\colon \mathbb{R} \to \mathbb{R}^+$ is onto. Define $g\colon \mathbb{R} \to \mathbb{R}^+$ by $g(x) = (f(x))^2$. Prove that g is onto.

15. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is one-to-one and let $a, b \in \mathbb{R}$ where $a \neq 0$. Define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) = af(x) + b$. Prove that g is one-to-one.

16. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is onto and let $a, b \in \mathbb{R}$ where $a \neq 0$. Define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) = af(x) + b$. Prove that g is onto.

17. Define $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $f(m,n) = 2^m 3^n$. Prove that f is one-to-one.

18. Let $f\colon \mathbb{R} \to \mathbb{R}$ be as in Example 2. Prove that f is one-to-one and onto.

Exercise Notes: For Exercise 11, if $x^2 = y^2$ then $|x| = |y|$. For Exercise 17, let $m, n, i, j \in \mathbb{N}$. Assume $f(m,n) = f(i,j)$. Prove $m = i$ and $n = j$.

6.3 Composition of Functions

If the domain of a function equals the co-domain of another function, then we can use these two functions to construct a new function called the *composite function*. The composite function is defined by taking the output of one these functions and using that as the input for the other function. The formal mathematical definition appears below.

Definition 6.3.1. For functions $g\colon A \to B$ and $f\colon B \to C$, one forms the **composite function** $(f \circ g)\colon A \to C$ by defining $(f \circ g)(x) = f(g(x))$ for all $x \in A$.

For example, let $g\colon A \to B$ and $f\colon B \to C$ be the functions in Fig. 6.7. An arrow diagram for the composite function $(f \circ g)\colon A \to C$ appears in Fig. 6.8.

Example 1. Let $g\colon \mathbb{R} \to \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ be the functions defined by $f(x) = \frac{1}{x^2+2}$ and $g(x) = 2x - 1$. Find formulas for $(f \circ g)(x)$ and $(g \circ f)(x)$. Is $f \circ g = g \circ f$?

Solution. Let $x \in \mathbb{R}$. We evaluate the function $(f \circ g)(x)$ as follows:

$$(f \circ g)(x) = f(g(x)) = f(2x-1) = \frac{1}{(2x-1)^2 + 2}.$$