Proposition 6.1.11. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = (x-1)^2$ and let $g\colon \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2 - 2x + 1$. Prove that $f = g$.*

Proof. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be defined as stated in the proposition. Let $x \in \mathbb{R}$. We prove that $f(x) = g(x)$ as follows:

$$\begin{aligned} f(x) &= (x-1)^2 && \text{by the definition of } f \\ &= x^2 - 2x + 1 && \text{by algebra} \\ &= g(x) && \text{by definition of } g. \end{aligned}$$

Therefore, $f = g$. □

Proposition 6.1.12. *Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be functions. Define $s\colon \mathbb{R} \to \mathbb{R}$ and $t\colon \mathbb{R} \to \mathbb{R}$ by*

$$s(x) = f(x) \cdot g(x) \text{ for all } x \in \mathbb{R} \tag{6.8}$$

$$t(x) = g(x) \cdot f(x) \text{ for all } x \in \mathbb{R}. \tag{6.9}$$

Then $s = t$.

Proof. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be functions and let $s\colon \mathbb{R} \to \mathbb{R}$ and $t\colon \mathbb{R} \to \mathbb{R}$ be defined by (6.8) and (6.9). Let x be a real number. We shall prove that $s(x) = t(x)$ as follows:

$$\begin{aligned} s(x) &= f(x) \cdot g(x) && \text{by (6.8)} \\ &= g(x) \cdot f(x) && \text{by the commutative law of multiplication} \\ &= t(x) && \text{by (6.9).} \end{aligned}$$

Therefore, $s = t$. □

Remark 6.1.13. Given functions $f\colon A \to B$ and $g\colon A \to B$, to show that $f \neq g$ you must find at least one element $x \in A$ and show that $f(x) \neq g(x)$.

Exercises 6.1

1. Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = 4n + 1$. Determine the range of f.

2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = -x^2 + 4x$. Determine the range of f.

3. Consider the functions $f\colon \mathbb{R}^+ \to \mathbb{R}$ and $g\colon \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = \frac{16x^2-1}{4x+1}$ and $g(x) = 4x - 1$ for all $x \in \mathbb{R}^+$. Prove that $f = g$.

4. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be functions. Define $s\colon \mathbb{R} \to \mathbb{R}$ and $t\colon \mathbb{R} \to \mathbb{R}$ by

$$s(x) = 2f(x) + 3g(x) \quad \text{for all } x \in \mathbb{R} \tag{6.10}$$

$$t(x) = 6f(x) - g(x) \quad \text{for all } x \in \mathbb{R}. \tag{6.11}$$

Prove that if $s = t$, then $f = g$.