6. Let $I = \{i \in \mathbb{R} : 1 \le i\} = [1, \infty)$ and let $A_i = \left\{x \in \mathbb{R} : -\frac{1}{i} \le x \le 2 - \frac{1}{i}\right\}$, for each $i \in I$. Express $\bigcup_{i \in I} A_i$ and $\bigcap_{i \in I} A_i$ in interval notation, if possible.

7. Let p and q be prime numbers. Define $A = \{p^i : i \in \mathbb{N}\}$ and $B = \{q^i : i \in \mathbb{N}\}$. Prove that if $A \cap B \neq \varnothing$, then $A = B$.

8. In our proof of Theorem 5.3.6(1) we applied the "double-subset" Proof Strategy 5.2.5(a). Reprove Theorem 5.3.6(1) using the "iff" Strategy 5.2.5(b).

9. Prove the following theorems:

(a) **Theorem.** Let $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ be indexed families of sets with the same indexed set I. Suppose $A_i \subseteq B_i$ for all $i \in I$. Then $\bigcup_{i \in I} A_i \subseteq \bigcup_{i \in I} B_i$.

(b) **Theorem.** Let $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ be indexed families of sets with the same indexed set I. Suppose $A_i \subseteq B_i$ for all $i \in I$. Then $\bigcap_{i \in I} A_i \subseteq \bigcap_{i \in I} B_i$.

(c) **Theorem.** Let $\{A_i : i \in I\}$ and $\{B_j : j \in J\}$ be indexed families of sets. If there is an $i_0 \in I$ such that $A_{i_0} \subseteq B_j$ for all $j \in J$, then $\bigcap_{i \in I} A_i \subseteq \bigcap_{j \in J} B_j$.

(d) **Theorem.** Suppose that A is a set and that $\{B_i : i \in I\}$ is an indexed family of sets. Then $A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} (A \cap B_i)$.

(e) **Theorem.** Suppose that A is a set and that $\{B_i : i \in I\}$ is an indexed family of sets. Then $A \cup \bigcap_{i \in I} B_i = \bigcap_{i \in I} (A \cup B_i)$.

(f) **Theorem.** Suppose that A is a set and that $\{B_i : i \in I\}$ is an indexed family of sets. Then $A \setminus \bigcap_{i \in I} B_i = \bigcup_{i \in I} (A \setminus B_i)$.

10. Let $\{B_x : x \in \mathbb{R}^+\}$ be the family of sets in Example 3 on page 157. Evaluate $\bigcap_{x \in \mathbb{R}^+} B_x$ and $\bigcup_{x \in \mathbb{R}^+} B_x$.

11. Let $\{B_i : i \in I\}$ be the family of sets in Example 4. Evaluate $\bigcap_{i \in I} B_i$ and $\bigcup_{i \in I} B_i$.

12. Prove Theorem 5.3.7.

13. Let $\mathcal{F}$ and $\mathcal{G}$ be two families of sets. Prove that $\bigcup(\mathcal{F} \cup \mathcal{G}) = (\bigcup \mathcal{F}) \cup (\bigcup \mathcal{G})$.

5.4 The Axioms of Set Theory

Albert Einstein devoted much of his professional life to the search for a unified theory of physics, that is, a theory that fully explains and links together all known physical phenomena. Einstein was not successful in his quest to find such a theory. Since then one of the most engaging goals for researchers in physics has been to construct a unifying theory for physics. Stephen Hawking concludes his book *A Brief History of Time* with the hope that someone will discover a unified theory and observes that if such a theory can be realized, then "it would be the ultimate triumph – for then we would know the mind of God."