and $B = \{2,3\}$. Clearly, the set $X = \{1,3\}$ is subset of $A \cup B$ and thus, $X \in \mathcal{P}(A \cup B)$. Since X is not a subset A and is also not a subset of B, we see that $X \notin \mathcal{P}(A) \cup \mathcal{P}(B)$. So $X \in \mathcal{P}(A \cup B)$ and $X \notin \mathcal{P}(A) \cup \mathcal{P}(B)$. Therefore, $\mathcal{P}(A \cup B) \neq \mathcal{P}(A) \cup \mathcal{P}(B)$.

Exercises 5.2

Prove the following theorems, where A, B, C, and D are sets.

1. **Theorem.** If $A \subseteq B$, then $A \subseteq A \cup B$ and $A \cap B \subseteq A$.
2. **Theorem.** If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
3. **Theorem.** $C \subseteq A$ and $C \subseteq B$ if and only if $C \subseteq A \cap B$.
4. **Theorem.** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
5. **Theorem.** $(A \setminus B) \cap (C \setminus B) = (A \cap C) \setminus B$.
6. **Theorem.** $A \cap (B \cap C) = (A \cap B) \cap C$ and $A \cup (B \cup C) = (A \cup B) \cup C$.
7. **Theorem.** $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$.
8. **Theorem.** If $A \setminus B \subseteq C$, then $A \setminus C \subseteq B$.
9. **Theorem.** If $A \subseteq B$ and $B \cap C = \varnothing$, then $A \subseteq B \setminus C$.
10. **Theorem.** If $A \setminus B \subseteq C$ and $A \nsubseteq C$, then $A \cap B \neq \varnothing$.
11. **Theorem.** $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.
12. **Theorem.** $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cup D)$.
13. **Theorem.** $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.
14. **Theorem.** $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.
15. **Theorem.** $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

Exercise Notes: For Exercises 4–6: Use Proof Strategy 5.2.5(b) and review the propositional logic laws in Section 1.1.5. For Exercise 7, one may want to use Proof Strategy 5.2.5(a). For Exercise 8, to prove that $x \in B$, use proof by contradiction. For Exercise 10, review Remark 5.1.2(2).

5.3 Indexed Families of Sets

Given a property $P(x)$ we can form the truth set $\{x : P(x)\}$ when the universe is understood. There is another way to build sets. For example, consider the set S of all perfect squares, that is, the set of all numbers of the form n^2 for some natural number n. We can define S in two ways:

1. $S = \{x : (\exists n \in \mathbb{N})(x = n^2)\} = \{1,4,9,16,25,\ldots\}$.
2. $S = \{n^2 : n \in \mathbb{N}\} = \{1,4,9,16,25,\ldots\}$.