Exercises 4.4

1. Prove, for every integer $n \geq 1$, that $1+3+5+\cdots+(2n-1)=n^2$.
2. Prove, for every integer $n \geq 0$, that $1+2+2^2+\cdots+2^n=2^{n+1}-1$.
3. Prove that $2+6+18+\cdots+2\cdot 3^{n-1}=3^n-1$, for every integer $n \geq 1$.
4. Prove that $\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$, for all integers $n \geq 2$.
5. Prove that $\sum_{k=1}^{n}(2^k-2^{k-1})=2^n-1$ for all integers $n \geq 1$, by mathematical induction.
6. Prove that $\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$ holds, for every integer $n \geq 1$.
7. Let c be a real number and let $a_m, a_{m+1}, \ldots$ be a sequence where m is an integer. Prove that $\sum_{k=m}^{n}ca_k=c\sum_{k=m}^{n}a_k$, for all $n \geq m$.
8. Prove, for every integer $n \geq 1$, that $\sum_{k=1}^{n}\frac{1}{4k^2-1}=\frac{n}{2n+1}$.
9. Prove that $\sum_{k=1}^{n}k\cdot k!=(n+1)!-1$, for all integers $n \geq 1$.
10. Prove that $\sum_{k=1}^{n}(-1)^k k^2=(-1)^n\frac{n(n+1)}{2}$, for all integers $n \geq 1$.
11. Prove, for every integer $n \geq 1$, that $\sum_{k=1}^{n}k^3=\left[\frac{n(n+1)}{2}\right]^2$.
12. Prove for all integers $n \geq 1$ that $\binom{n}{k}$ is a natural number whenever k is an integer satisfying $0 \leq k \leq n$.
13. Find a closed-form solution for the sum $\sum_{k=0}^{n}(2^k-1)(3^k+1)$.
14. Prove that $\sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)}=\frac{n}{2n+1}$, for all integers $n \geq 1$.
15. Let $r \neq 1$ be nonzero. Find a closed-form solution for the sum $\sum_{k=-9}^{n}r^k$.
16. (The Binomial Theorem) Let x and y be variables, representing real numbers. Prove for every integer $n \geq 1$ that $\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k=(x+y)^n$.

Exercise Notes: For Exercise 8, note that $4(n+1)^2-1=(2n+1)(2n+3)$. For Exercise 12, use induction on n and Exercise 19 on page 116. For Exercise 16, in the inductive step use Exercise 20 on page 116.