$$= (n^2 + n + 1) + 2(n + 1) \quad \text{by distributivity}$$
$$= 2i + 2(n + 1) \quad \text{by induction hypothesis (IH)}$$
$$= 2(i + n + 1) \quad \text{by distributivity.}$$

Hence, $(n+1)^2 + (n+1) + 2 = 2j$ where $j = i + n + 1$ is an integer. Therefore, $(n+1)^2 + (n+1) + 2$ is even and this completes the proof. □

Exercises 4.2

1. Using mathematical induction, prove that for every integer $n \geq 0$, the number $n^2 + 3n + 5$ is odd.
2. Prove, by induction, that for every integer $n \geq 0$, either n is even or n is odd.
3. Using Exercise 2, prove that every integer n is either even or odd.
4. Using Exercise 3, prove that for every integer n we have that $n(n+1)$ is even.
5. For any integer $n \geq 1$, let $P(n)$ be the statement: $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. What is statement $P(3)$? What is statement $P(1)$? What is statement $P(n+1)$?
6. Prove that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all integers $n \geq 1$, by induction.

Exercise Notes: For Exercise 2, use the induction proof strategy

Base step: Prove 0 is even or odd.
Inductive step: Let $n \geq 0$ be an integer.
Assume n is even or n is odd.
Prove $n + 1$ is even or $n + 1$ is odd.

In the inductive step, since you are assuming an 'or' statement, use a proof by cases. For Exercise 3, there are two cases: (1) $n \geq 0$; (2) $n < 0$. For Exercise 4, there are two cases: (1) n is odd; (2) n is even.

4.3 Sequences, Sums, and Factorials

In mathematics one often works with regular patterns or repeated processes. The main tool used to study repeated processes is the *sequence*, a fundamental concept with a rich history in mathematics. Sequences are interesting mathematical objects with lots of surprising properties, many of which can be verified by mathematical induction.