The proof of Theorem 4.1.8 shows that when you have a list ($\star$) $p_1, p_2, p_3, \ldots, p_n$ of the first n primes, then any prime number that evenly divides $p_1 p_2 p_3 \cdots p_n + 1$ is not in the list ($\star$). We illustrate this result with an example using the first six primes. Observe that $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30{,}031$ which is divisible only by the two primes 59 and 509, both of which are not in the list $2, 3, 5, 7, 11, 13$.

Exercises 4.1

1. Explain why Theorem 4.1.2 implies the Well-Ordering Principle 4.1.1.
2. Let ($*$) $q_1, q_2, \ldots, q_m$ be *any* finite list of prime numbers. Let p be a prime that evenly divides $q_1 q_2 \cdots q_m + 1$. Prove that p is not in the list ($*$).
3. Prove the statement: For all integers $n \geq 1$, we can write $n = 2^k \cdot m$ for some integers k and m where $k \geq 0$ and m is odd.
4. Prove that for every integer $n \geq 0$, either n is even or n is odd.
5. Prove that for every integer $n \geq 3$, we have $2^n \geq 2n + 1$.
6. Let $P(n)$ be a statement that is defined for all integers $n \geq 1$. Suppose the following two conditions hold: (a) $P(1)$ is true, and (b) for all integers $n \geq 1$, if $P(n)$ holds then $P(n+1)$ also holds. Use the Well-Ordering Principle 4.1.1 to prove that $P(n)$ must be true for all integers $n \geq 1$.

Exercise Notes: For Exercise 3, use the well-ordering Proof Strategy 4.1.5 where $P(n)$ is the statement "$n = 2^k \cdot m$ for some integers k and m where $k \geq 0$ and m is odd." There are two cases to consider about N. If N is even, then $N = 2i$ for some integer i where $1 \leq i < N$. If N is odd, then note that $N = 2^0 N$. For Exercise 4, use Proof Strategy 4.1.5. Show that $N > 0$. So, $0 \leq N - 1 < N$ and thus $N - 1$ is either even or odd. For Exercise 5, using Proof Strategy 4.1.5, one obtains the assumption $2^N < 2N + 1$. Observe that $N > 3$ and thus, $3 \leq N - 1 < N$. So, you can conclude that $2^{N-1} \geq 2(N-1) + 1$. Multiply both sides of this latter inequality by 2 to derive a contradiction. For Exercise 6, if $N \geq 1$ is the least such integer satisfying $\neg P(N)$, then explain why $N = n + 1$ for some $n \geq 1$ and observe that $n < N$.

4.2 Proof by Mathematical Induction

Mathematical induction is a powerful method for proving theorems about the natural numbers. Suppose you have a statement $P(n)$ that you want to prove is true for every integer n greater than or equal to the integer b. How can you prove this statement by mathematical induction? First you prove that the statement definitely holds for b. Then you have to prove that whenever the statement holds for an integer $n \geq b$,