CASE 3: Assume $x = 3q + 2$. Then

$$x^2 = (3q+2)^2 = 9q^2 + 12q + 4 = 9q^2 + 12q + 3 + 1 = 3\left(3q^2 + 4q + 1\right) + 1.$$

Therefore, $x^2 = 3k + 1$ where $k = 3q^2 + 4q + 1$ is an integer. □

To use a division by cases, in a proof, you must first identify all the possibilities, that is, all the cases. Then you must prove the conclusion under each case. The definition of absolute value, below, is one that is given in most calculus books. Observe that the definition of $|x|$ is based on two cases; namely, $x \geq 0$ and $x < 0$. Consequently, proofs about the absolute value function often use a division by these two cases.

Definition 3.6.6. Given a real number x, the **absolute value** of x, denoted by $|x|$, is defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0. \end{cases}$$

One of the most commonly used functions in mathematics is the absolute value function. The absolute value $|x|$ is simply the distance from x to the origin. In more advanced mathematics, especially in real analysis, the absolute value function is used extensively (see Chapter 9 and Theorems 9.2.3–9.2.4).

Exercises 3.6

Prove the following theorems:

1. **Theorem.** If x is an integer, then x^2 has the form $4k$ or $4k+1$ for an integer k.
2. **Theorem.** Let n and m be integers. If mn is even, then m is even or n is even.
3. **Theorem.** Let n and m be integers. If $m+n$ is odd, then m is odd or n is odd.
4. **Theorem.** If $a > 0$ is a real number, then $1 < a + \frac{1}{a}$.
5. **Theorem.** Let a and b be real numbers. If $0 \leq a \leq b$, then $a^2 \leq b^2$.
6. **Theorem.** Let a, b, x, y be non-negative real numbers. If $a \leq b$ and $x \leq y$, then $ax \leq by$.
7. **Theorem.** Let n and d be integers where $d \geq 1$. There exists an integer k such that $n - dk \geq 0$.
8. **Theorem.** For all real numbers x we have that $x^2 \geq 0$.
9. **Theorem.** For all real numbers x and y, if $x \geq 2$ and $y \geq 2$, then $xy \geq x + y$
10. **Theorem.** Let x be a real number. Then $|x| \geq 0$.
11. **Theorem.** Let x be a real number. Then $x \leq |x|$.
12. **Theorem.** Let x, y be real numbers. Then $|xy| = |x|\,|y|$.
13. **Theorem.** Let x be a real number. Then $x^2 = |x|^2$.

Exercise Notes: For Exercise 1, use the fact that x is either even or odd. For Exercises 2 and 3, recall that every integer is either even or odd. For Exercise 4, since either $a < 1$ or $a = 1$ or $1 < a$, use a division by cases. More specifically, use the proof diagram

Assume $a > 0$ is a real number.
  Case 1: Assume $a < 1$.
    Prove $1 < a + \frac{1}{a}$.
  Case 2: Assume $a = 1$.
    Prove $1 < a + \frac{1}{a}$.
  Case 3: Assume $1 < a$.
    Prove $1 < a + \frac{1}{a}$.

Note: $0 < \frac{1}{a}$. For Exercise 5, there are four cases to consider: (1) $0 = a < b$; (2) $0 = a = b$; (3) $0 < a = b$; (4) $0 < a < b$. Recall Theorem 3.3.2. For Exercise 6, use a division by cases and Exercise 10 on page 84. For Exercise 7, there are two cases to consider: $n \leq 0$ and $n > 0$. For Exercise 9, consider the cases $x \leq y$ and $y \leq x$. Note that when $y \geq 2$ and $x > 0$, one can conclude that $xy \geq 2x = x + x$ (review the Laws of Inequality 3.1.5). For Exercises 10 and 11, there are two cases (1) $x \geq 0$; (2) $x < 0$. Also, note that if $x < 0$ then $|x| = -x$ by Definition 3.6.6. For Exercise 12, there are four cases (1) $x, y \geq 0$; (2) $x < 0$ and $y \geq 0$; (3) $x \geq 0$ and $y < 0$; (4) $x, y < 0$.

3.7 Statements of the Form $P \leftrightarrow Q$

Consider the biconditional statement "P if and only if Q." Suppose we want to show that such a statement is true. Recall that this biconditional is equivalent to the conjunction of the two conditional statements "if P then Q" and "if Q then P" (see the biconditional law on page 17). We arrive at the following proof strategy.

Proof Strategy 3.7.1. Given a diagram containing the form

Prove $P \leftrightarrow Q$

replace this form with

Prove $P \to Q$
Prove $Q \to P$.

In other words one has to prove $P \to Q$ and prove $Q \to P$, separately.

Theorem 3.7.2. *Let n be an integer. Then $6 \mid n$ if and only if $2 \mid n$ and $3 \mid n$.*

Proof. Let n be an integer. We will prove that $6 \mid n$ if and only if $2 \mid n$ and $3 \mid n$.

($\Rightarrow$). First we prove that if $6 \mid n$, then $2 \mid n$ and $3 \mid n$. Assume $6 \mid n$. Thus, $n = 6i$ for some $i \in \mathbb{Z}$. So, $n = 6i = 2(3i) = 3(2i)$ and hence, $2 \mid n$ and $3 \mid n$.