Exercises 3.5

Prove the following theorems:

1. **Theorem.** Let a, b be real numbers. If $a > 0$ and $b > 0$, then $(a+b)^2 > a^2 + b^2$.
2. **Theorem.** Let a, b be real numbers. If $a < 0$ and $b < 0$, then $(a+b)^2 > a^2 + b^2$.
3. **Theorem.** For all $x \in \mathbb{R}$ and $y \in \mathbb{R}$, if x and y are rational, then $x + y$ is rational.
4. **Theorem.** For all integers a, b, and c, if $c \mid a$ and $c \mid b$, then $c \mid (a+b)$, $c \mid (a-b)$, and $c \mid (ai)$ for any integer i.
5. **Theorem.** Let n be an integer. If $21 \mid n$, then $3 \mid n$ and $7 \mid n$.
6. **Theorem.** Suppose n is an integer. If $3 \mid n$ and $7 \mid n$, then $21 \mid n$.
7. **Theorem.** For every integer n, if n is odd, then $4 \mid (n^2 - 1)$.
8. **Theorem.** Suppose m, n are positive integers. If $m \mid n$, then $m \leq n$.
9. **Theorem.** For all positive integers a and b, if $a \mid b$ and $b \mid a$, then $a = b$.
10. **Theorem.** Let a, b, x, y be negative integers. If $a < b$ and $x < y$, then $ax > by$.
11. **Theorem.** Let $a > 0$ and $b < -4$ be real numbers. Then $ab + b < -4(a+1)$.
12. **Theorem.** For all integers a and b, if $a \mid b$, then $a^2 \mid b^2$.
13. **Theorem.** Suppose m, a, b, c, d are integers. If $m \mid (a-b)$ and $m \mid (c-d)$, then $m \mid ((a+c) - (b+d))$.
14. **Theorem.** Let m, a, b, c, d be integers. If $m \mid (a-b)$ and $m \mid (c-d)$, then $m \mid (ac - bd)$.
15. **Theorem.** Let a, b, d be real numbers. If $0 \leq a < d$ and $0 \leq b < d$, then $a - b < d$ and $b - a < d$.
16. **Theorem.** If $0 \leq a < d$ and $0 \leq b < d$, then $-d < a - b < d$ where $a, b, d \in \mathbb{R}$.
17. **Theorem.** For all integers a, b, c, d, if $a \neq c$ and $ad \neq bc$, then there exists a unique rational number x such that $\frac{ax+b}{cx+d} = 1$.

Exercise Notes: For Exercises 1 and 2, one should review the substitution properties of inequality 3.3.3. For Exercise 6, use the identity $n = 7n - 6n$. For Exercise 17, after you identify x in your proof, you must prove that $cx + d \neq 0$.

3.6 Statements of the Form $P \vee Q$

Consider the statement "P or Q." To show that this statement is true, we must verify that either P is true or that Q is true. So we can try to prove P or try to prove Q. This direct approach can sometimes be difficult, as we may then have to work with an inadequate set of assumptions. Fortunately, logic offers us an easier approach. We know that $(P \vee Q)$, $(\neg P \to Q)$, and $(\neg Q \to P)$ are all logically equivalent. Thus, to prove $(P \vee Q)$, we can either prove $(\neg P \to Q)$ or prove $(\neg Q \to P)$. In either case, we obtain a new assumption that we can use in our proof. We can now introduce a proof strategy for such "or" statements.