Theorem 3.4.18. *Let a, b be real numbers where $a \neq 0$. Then there exists a unique real number x satisfying $ax + b = 0$.*

Proof Analysis. We are given real numbers a and b with $a \neq 0$. First we need to prove that there exists a real number x that satisfies the equation $ax + b = 0$; that is, we need to prove that $(\exists x \in \mathbb{R})(ax + b = 0)$. Afterwards, we must prove that there is only one such solution. We use the uniqueness–Proof Strategy 3.4.17 to obtain the following proof diagram where $P(x)$ is the assertion that $ax + b = 0$:

Assume $a, b \in \mathbb{R}$ where $a \neq 0$.
Existence: Prove $(\exists x \in \mathbb{R})(ax + b = 0)$.
Uniqueness: Assume $(ax + b = 0) \wedge (ay + b = 0)$.
Prove $x = y$.

We apply the $\exists$-Proof Strategy 3.4.5(b) to obtain the diagram

Assume $a, b \in \mathbb{R}$ where $a \neq 0$.
Existence: Let $x =$ (the value in $\mathbb{R}$ you found).
Prove $ax + b = 0$.
Uniqueness: Assume $(ax + b = 0) \wedge (ay + b = 0)$.
Prove $x = y$.

To find this x, we simply solve the equation $ax + b = 0$ for x and obtain $x = -\frac{b}{a}$. We have our final proof diagram

Assume $a, b \in \mathbb{R}$ where $a \neq 0$.
Existence: Let $x = -\frac{b}{a}$.
Prove $ax + b = 0$.
Uniqueness: Assume $(ax + b = 0) \wedge (ay + b = 0)$.
Prove $x = y$.

This final diagram will guide our composition of the following proof. Ⓐ

Proof. Let a, b be real numbers where $a \neq 0$. First we prove that there exists a real number x satisfying $ax + b = 0$. Then we will prove that such an x is unique.

Existence: Let $x = -\frac{b}{a}$. Since $a \neq 0$, we see that x is a real number. Now, since $x = -\frac{b}{a}$, using a little algebra we get $ax + b = a(-\frac{b}{a}) + b = 0$. Therefore, there is an x satisfying $ax + b = 0$.

Uniqueness: Suppose (1) $ax + b = 0$ and (2) $ay + b = 0$. We prove that $x = y$. From (1) and (2) we see that $ax + b = ay + b$. Using algebra, we conclude that $x = y$. □

Exercises 3.4

Prove the following theorems:

1. **Theorem.** Let $c \neq 1$ be a real number. There exists a unique real number x satisfying $\frac{x+1}{x-2} = c$.

2. **Theorem.** Let m be an integer. If m is odd, then m^2 is odd.
3. **Theorem.** Let m be an integer. If m is even, then $m+5$ is odd.
4. **Theorem.** Let m and n be integers. If n is even, then mn is even.
5. **Theorem.** For all integers m and n, if $m-n$ is even, then m^2-n^2 is even.
6. **Theorem.** There exists a real number x such that $|3x-2| = -7x$.
7. **Theorem.** For every real number $a > 3$, there is a real number $x < 0$ such that $|3x-2| = -ax$.
8. **Theorem.** For every real number $y > 0$, there is a real number $x < 0$ such that $y^2 + 2xy = -x^2$.
9. **Theorem.** For each real number x, there is a real number y that satisfies the equation $y^2 - 2xy = 2$.
10. **Theorem.** There is a real number $d > 0$ such that for all real numbers x, if $|x-1| < d$, then $|3x-3| < \frac{1}{2}$.
11. **Theorem.** For every integer n, if n is odd, then $\frac{n-1}{2}$ is an integer.
12. **Theorem.** Let a,b be integers where $a \neq 0$ or $b \neq 0$. There is an integer $n \geq 1$ of the form $n = sa + tb$ for some integers s,t.
13. **Theorem.** For every integer i there is a unique integer j such that $3j + 9i = 6$.
14. **Theorem.** For every real number x there is a unique real number y such that $yx^2 - 3x = -2y$.
15. **Theorem.** There is a unique real number y such that $yx + 6 = 2x + 3y$ for every real number x.
16. **Theorem.** Let $c \leq 2$ be a real number. Suppose $x + y \leq xy$ for all real numbers x and y satisfying $x \geq c$ and $y \geq c$. Then $c = 2$.

Exercise Notes: For Exercises 6 and 7, recall that $|y| = \pm y$ (see Definition 3.6.6). For Exercise 15, to prove uniqueness, note that if the equation $yx + 6 = 2x + 3y$ holds for every real number x, then the equation holds for $x = 1$. For Exercise 16, note that $2 \geq c$ and $c \geq c$.

3.5 Statements of the Form $P \wedge Q$

Consider the statement "P and Q." To show that this statement is true, we must show that P is true and show that Q is also true. We can now introduce a proof strategy for such "and" statements.

Proof Strategy 3.5.1. Given a diagram containing the form

$$\text{Prove } P \wedge Q$$