Contrapositive Law

1. $(P \to Q) \Leftrightarrow (\neg Q \to \neg P)$.

Biconditional Law

1. $(P \leftrightarrow Q) \Leftrightarrow (P \to Q) \wedge (Q \to P)$.

We can now use the above propositional logic laws, together with those listed in Section 1.1.5, to derive new logic laws.

Example 2. Show that $(P \to R) \wedge (Q \to R) \Leftrightarrow (P \vee Q) \to R$, by using propositional logic laws.

Solution. We first start with the more complicated side $(P \to R) \wedge (Q \to R)$ and derive the simpler side as follows:

$$\begin{aligned}
(P \to R) \wedge (Q \to R) &\Leftrightarrow (\neg P \vee R) \wedge (\neg Q \vee R) && \text{by Conditional Law(1)} \\
&\Leftrightarrow (\neg P \wedge \neg Q) \vee R && \text{by Distributive Law(4)} \\
&\Leftrightarrow \neg(P \vee Q) \vee R && \text{by De Morgan's Law(1)} \\
&\Leftrightarrow (P \vee Q) \to R && \text{by Conditional Law(1).}
\end{aligned}$$

Therefore, $(P \to R) \wedge (Q \to R) \Leftrightarrow (P \vee Q) \to R$. Ⓢ

Using a list of propositional components $A, B, C, \ldots$ and the logical connectives $\wedge, \vee, \neg, \to, \leftrightarrow$, we can form a variety of propositional sentences. For example,

$$(P \to R) \wedge \neg(Q \leftrightarrow (S \vee T)).$$

These connectives are also used to tie together a variety of mathematical statements. A good understanding of these logical connectives will allow us to more easily understand and construct mathematical proofs.

Exercises 1.2

1. Using truth tables, show that $\neg(P \to Q) \Leftrightarrow (P \wedge \neg Q)$.
2. Construct truth tables to show that $(P \leftrightarrow Q) \Leftrightarrow (P \to Q) \wedge (Q \to P)$.
3. Using truth tables, show that $P \Leftrightarrow (\neg P \to (Q \wedge \neg Q))$.
4. Which of the following statements are true and which are false?

(a) $(\pi^2 > 9) \to (\pi > 3)$.
(b) If $3 \geq 2$, then $3 \geq 1$.
(c) If $3 \geq 2$, then $3 \leq 1$.
(d) If $1 \geq 2$, then $1 \geq 1$.
(e) $(1 + 5 = 2) \to$ (the author is a genius).

(f) $(1+5=2) \to$ (the author is an idiot).
(g) $(\sin(2\pi) > 9) \to (\sin(2\pi) < 0)$.
(h) $3 \geq 2$ if and only if $3+5 \geq 7$.
(i) $1 \geq 2$ if and only if $1+5 \geq 7$.

5. Let C, W and R represent the propositions:

C : "The picnic has been canceled."
W : "It is windy."
R : "It is raining."

Using the propositions C, W, R analyze the logical forms of the following three statements, that is, write each statement symbolically. Then determine which of the statements are logically equivalent (justify your answers).

(a) If it is either windy or raining, then the picnic has been canceled.
(b) If the picnic has not been canceled, then it's not windy and it's not raining.
(c) The picnic has been canceled only if it's either windy or raining.

Now form the converse of each of the logical forms you obtained and then express each result in English.

6. Consider the propositions:

P : "Pigs fly."
S : "The sky is polluted with pies."
M : "Math is a favorite subject."
F : "Food is in short supply."

Translate each of the following propositional sentences into English sentences:

(a) $\neg P \to (S \vee F)$
(b) $M \vee (S \wedge \neg F)$
(c) $P \to (M \to S)$
(d) $(P \to M) \to S$
(e) $(F \wedge S) \leftrightarrow P$
(f) $(F \wedge \neg S) \to (\neg P \vee M)$
(g) $\neg F \to (\neg M \leftrightarrow (P \vee S))$.

7. Using truth tables, show that $(P \vee Q)$ and $(\neg Q \to P)$ are logically equivalent.
8. Using propositional logic laws, show that $(P \to R) \wedge (Q \to R) \Leftrightarrow (P \vee Q) \to R$.
9. Using propositional logic laws, show that $(P \to R) \vee (Q \to R) \Leftrightarrow (P \wedge Q) \to R$.
10. Using propositional logic laws, show that $P \to (Q \to R) \Leftrightarrow (P \wedge Q) \to R$.
11. Show that $(P \to Q) \to R$ and $P \to (Q \to R)$ are not logically equivalent.
12. **True or False:** The negation of the statement *"If Sue is Pina's daughter, then Diane is Sue's cousin"* is logically equivalent to the assertion *"If Sue is Pina's daughter, then Diane is not Sue's cousin."*
13. Write a negation, in English, of the statement: *If n is an even number, then n is not odd.*