Example 7. Using logic laws, find a simpler sentence equivalent to the formula $\neg Q \vee \neg(\neg P \vee \neg Q)$.

Solution. We start with $\neg Q \vee \neg(\neg P \vee \neg Q)$ and apply logic laws as follows:

$$\begin{aligned}
\neg Q \vee \neg(\neg P \vee \neg Q) &\Leftrightarrow \neg Q \vee (\neg\neg P \wedge \neg\neg Q) && \text{by De Morgan's Law} \\
&\Leftrightarrow \neg Q \vee (P \wedge Q) && \text{by Double Negation Law} \\
&\Leftrightarrow (\neg Q \vee P) \wedge (\neg Q \vee Q) && \text{by Distributive Law} \\
&\Leftrightarrow (\neg Q \vee P) && \text{by Tautology Law.}
\end{aligned}$$

Therefore, $\neg Q \vee \neg(\neg P \vee \neg Q) \Leftrightarrow \neg Q \vee P$. Thus, $\neg Q \vee P$ is a simplified version of $\neg Q \vee \neg(\neg P \vee \neg Q)$. Ⓢ

Example 8. Using propositional logic laws, show that $\neg(P \wedge \neg Q) \Leftrightarrow (\neg P \vee Q)$.

Solution. We start with the more complicated side $\neg(P \wedge \neg Q)$ and derive the simpler side as follows:

$$\begin{aligned}
\neg(P \wedge \neg Q) &\Leftrightarrow (\neg P \vee \neg\neg Q) && \text{by De Morgan's Law} \\
&\Leftrightarrow (\neg P \vee Q) && \text{by Double Negation Law.}
\end{aligned}$$

Therefore, $\neg(P \wedge \neg Q) \Leftrightarrow (\neg P \vee Q)$. Ⓢ

Exercises 1.1

1. Only one of the following is a tautology. Which one is it?

(a) $(P \vee \neg P) \wedge Q$.
(b) $(P \vee \neg P) \vee Q$.

2. Using one of De Morgan's Laws, write a negation of the statement: *Ron runs on Thursdays and Pete plays poker on Saturdays.* Express you answer in English.

3. Use one of De Morgan's Laws to write a negation, in English, of the statement: *My computer program has an error or the wrong value is assigned to a constant.*

4. Using propositional logic laws (see Section 1.1.5), supply a law justifying each step:

$$\begin{aligned}
(P \vee \neg Q) \wedge (\neg P \vee \neg Q) &\Leftrightarrow (\neg Q \vee P) \wedge (\neg Q \vee \neg P) && \text{by } \underline{\hspace{3cm}} \\
&\Leftrightarrow \neg Q \vee (\neg P \wedge P) && \text{by } \underline{\hspace{3cm}} \\
&\Leftrightarrow \neg Q && \text{by } \underline{\hspace{3cm}}
\end{aligned}$$

5. Using the propositional logic laws in Section 1.1.5, find simpler sentences (see Example 7) that are equivalent to the following:

(a) $\neg(\neg P \wedge \neg Q)$.
(b) $\neg Q \wedge \neg(\neg P \wedge \neg Q)$.
(c) $\neg(\neg P \vee Q) \vee (P \wedge \neg R)$.

6. Which of the following statements are true and which are false?

(a) $(\pi^2 > 9) \wedge (\pi > 3)$.
(b) $(\pi^2 > 9) \vee (\pi > 3)$.
(c) $(\sin(2\pi) > 9) \vee (\sin(2\pi) < 0)$.
(d) $(\sin(\pi) > 9) \vee \neg(\sin(\pi) \leq 0)$.

7. Using truth tables, show that $(\neg P \vee Q) \vee (P \wedge \neg Q)$ is a tautology. What can you conclude about the sentence $\neg((\neg P \vee Q) \vee (P \wedge \neg Q))$?
8. Using propositional logic laws, show that $P \vee (Q \wedge \neg P) \Leftrightarrow P \vee Q$.
9. Using logic laws, show that $\neg(P \vee \neg Q) \vee (\neg P \wedge \neg Q) \Leftrightarrow \neg P$.

1.2 The Conditional and Biconditional Connectives

1.2.1 Conditional Statements

Many mathematical theorems have the form "if P, then Q" or, equivalently, "P implies Q." Here is one important example that you may have seen in your calculus course:

Theorem. *If f is differentiable at the point a, then f is continuous at a.*

Let D be the proposition "f is differentiable at the point a" and let C be the proposition "f is continuous at a." The theorem can now be expressed as

Theorem. *If D, then C.*

A *conditional statement* has the form "if P, then Q." The statement P is called the **hypothesis** and the statement Q is called the **conclusion**. Thus, a conditional statement asserts that the truth of the hypothesis "implies" the truth of the conclusion. This is such an important idea in mathematics that we will now introduce a logical connective which will capture the mathematical notion that the hypothesis implies the conclusion.

The Conditional Connective. Given propositions P and Q, the conditional connective $\to$ means "implies" and can be used to form the sentence $P \to Q$. The sentence $P \to Q$ can be read as "P implies Q" or "if P, then Q."