> This revelation was a scientific event of the highest importance. Quite possibly it marked the origin of what we consider the specifically Greek contribution to rigorous procedure [mathematical proof] in mathematics. Certainly it has profoundly affected mathematics and philosophy from the time of the Greeks to the present day.

Since $(\sqrt{2})^2 = 2$, Theorem 3.8.8 implies that $\sqrt{2}$ is irrational. Thus, we now know that there is at least one irrational number. Are there any others? Yes, there are infinitely many irrational numbers. In fact, in Section 6.5 it will be shown that there are more irrational numbers than there are rational ones (see Exercise 18 on page 206). Consequently, one can prove that most irrational numbers cannot be obtained by performing algebraic operations on rational numbers. In particular, the vast majority of irrational numbers cannot be realized by taking the square root of a rational number.

Exercises 3.8

Prove the following theorems and corollaries. A corollary is a statement that follows from a previously established theorem. Each corollary below follows from the theorem above it.

1. **Theorem.** Let x and y be real numbers. If $x^2 = y^2$, then $|x| = |y|$.
2. **Theorem.** Let a and b be natural numbers. If $ab = 1$, then $a = 1$ and $b = 1$.
3. **Corollary.** Let a and b be natural numbers. If $a \mid b$ and $b \mid a$, then $a = b$.
4. **Corollary.** Let n be an integer. Then n is not both even and odd.
5. **Theorem.** Let r and $s \neq 0$ be rational numbers. If x is irrational, then $r + sx$ is irrational.
6. **Theorem.** Let $a > 0$ be a real number. Then $\frac{1}{a} > 0$.
7. **Corollary.** Let x and y be real numbers where $x > 0$. If $xy > 0$, then $y > 0$.
8. **Corollary.** Suppose a, b, c, d are all positive real numbers. If $ab = cd$ and $a \leq c$, then $d \leq b$.
9. **Corollary.** Suppose a, b, c are positive real numbers. If $a < c$, then $\frac{a}{b} < \frac{c}{b}$.
10. **Corollary.** Let a, b, d be positive real numbers. If $d < b$, then $\frac{a}{b} < \frac{a}{d}$.
11. **Corollary.** Let a, b, d be positive real numbers. If $a < c$ and $d < b$, then $\frac{a}{b} < \frac{c}{d}$.
12. **Theorem.** For all real numbers a and b, if $a^2 < b^2$, then $|a| < |b|$.
13. **Theorem.** For all real numbers x, if $x > 1$ then $0 < \frac{1}{x} < 1$.
14. **Theorem.** Let $x > 0$ be a real number. If x is irrational, then $\sqrt{x}$ is irrational.
15. **Theorem.** The real number $\sqrt{2} + \sqrt{3}$ is an irrational number.
16. **Theorem.** Let a and b be positive real numbers. Then $\sqrt{a} + \sqrt{b} > \sqrt{a + b}$.
17. **Theorem.** Let m and n be integers. Then mn is even if and only if m is even or n is even.
18. **Theorem.** Let a, b, c be integers. If $a + b = c$, then at least one of a, b, and c must be even.

19. Theorem. Let x and y be positive real numbers. Then $\frac{x}{y} + \frac{y}{x} \geq 2$.

20. Theorem. Let x and y be positive real numbers. Then $\frac{x+y}{2} \geq \sqrt{xy}$.

21. Theorem (Triangle Inequality). Let x, y be real numbers. Then $|x+y| \leq |x| + |y|$.

22. Theorem. Let x be a real number. Then $x^2 < 1$ if and only if $-1 < x < 1$.

Exercise Notes: For Exercise 4, if $2i = 2j + 1$ and $i, j \in \mathbb{Z}$, then $2(i - j) = 1$. For Exercise 12, see Exercise 13 on page 88. For Exercise 15, assume $\sqrt{2} + \sqrt{3} = \frac{a}{b}$ for nonzero integers a and b. Solve for $\sqrt{3}$ and then square both sides to obtain a contradiction. For Exercises 16 and 19, use contradiction. For Exercise 20, use contradiction and Theorem 3.3.2. For Exercise 21, use contradiction, Theorem 3.3.2 and Exercises 10, 11 and 13 of Section 3.6. For Exercise 22, see Exercise 3 on page 70 and Theorem 3.3.2.