4. *If* $a < b$ *and* $c > 0$*, then* $ac < bc$.
5. *If* $a < b$ *and* $c < 0$*, then* $ac > bc$.

We write $a > b$ when $b < a$, and the inequality $a \leq b$ means that $a < b$ or $a = b$. Similarly, $a \geq b$ means that $a > b$ or $a = b$. The Trichotomy Law allows us to assert that if $a \not< b$, then $a \geq b$. It should be noted that one can actually prove law 5 from laws 1–4. Furthermore, using the above laws of inequality, one can prove that $0 < 1$ and $-1 < 0$ (see Proposition 9.1.5 on page 296).

Theorem 3.1.6. *Let* a,b,c *be real numbers where* $a < b$*. Then* $a - c < b - c$.

Proof. Let a,b,c be real numbers where $a < b$. From law 3 of 3.1.5, we obtain the inequality $a + (-c) < b + (-c)$. Thus, we infer that $a - c < b - c$. □

Theorem 3.1.7. *Let* x *be a real number such that* $x > 2$*. Then* $x^2 > x + 1$.

Proof. Let x be a real number satisfying $(\star)$ $x > 2$. We shall prove that $x^2 > x + 1$. Since $x > 2$, we see that $x > 0$. From $(\star)$ and law 4 of 3.1.5, we conclude that $xx > 2x$. Hence $x^2 > 2x$ and so, $x^2 > x + x$. Because $x > 1$, we obtain $x + x > x + 1$ using law 3 of 3.1.5. Therefore, $x^2 > x + 1$. □

Theorem 3.1.8. *Let* a,b,c,d *be real numbers and suppose that* $a < b$ *and* $c < d$*. Then* $a + c < b + d$.

Proof. Let a,b,c,d be real numbers satisfying (1) $a < b$ and (2) $c < d$. We shall prove that $a + c < b + d$. From (1) and law 3 of 3.1.5, we see that $a + c < b + c$. From (2) and law 3 again, we have that $b + c < b + d$. So, $a + c < b + c < b + d$. Therefore, $a + c < b + d$. □

Theorem 3.1.9. *Let* a,b,c,d *be positive real numbers satisfying* $a < b$ *and* $c < d$*. Then* $ac < bd$.

Proof. Let a,b,c,d be positive real numbers satisfying (1) $a < b$ and (2) $c < d$. We shall prove that $ac < bd$. From (1) and law 4 of the laws of inequality 3.1.5, we conclude that $ac < bc$ because $c > 0$. From (2) and law 4 again, we derive the inequality $bc < bd$ because $b > 0$. So, $ac < bc < bd$. Therefore, $ac < bd$. □

Exercises 3.1

1. Let x and y be real numbers. Prove that $(x - y)(x^2 + xy + y^2) = x^3 - y^3$.
2. Let x and y be real numbers. Prove that $(x + y)(x^2 - xy + y^2) = x^3 + y^3$.
3. Let x and y be real numbers. Prove that $(x + y)^2 = x^2 + 2xy + y^2$.
4. Let x and y be real numbers. Prove that $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$, using Exercise 3.

5. Let $\varphi > 0$ be a real number satisfying $\varphi^2 - \varphi - 1 = 0$. Prove that $\varphi = \frac{1}{\varphi - 1}$.[2]
6. Let x be a real number such that $x > 1$. Prove that $x^2 > x$.
7. Let x be a real number where $x < 0$. Prove that $x^2 > 0$.
8. Let x be a real number where $x > 0$. Prove that $x^2 > 0$.
9. Let x be a real number where $x \neq 0$. Using Exercises 7 and 8, prove that $x^2 > 0$.
10. Let a and b be distinct real numbers. Using Exercise 9 prove that $a^2 + b^2 > 2ab$.
11. Let x and y be positive real numbers such that $x \neq y$. Using Exercise 10 prove that $\frac{x}{y} + \frac{y}{x} > 2$.
12. Let x be a real number such that $x^2 > x$. Must we conclude that $x > 1$?
13. Let x be a real number satisfying $0 < x < 1$. Prove that $x^2 < x$.
14. Let x be a real number where $x^2 < x$. Must we infer that $0 < x < 1$?
15. Let a and b are real numbers where $a < b$. Prove that $-a > -b$.
16. Let a, b be positive real numbers and let c, d be negative real numbers. Suppose $a < b$ and $c < d$. Prove that $ad > bc$.
17. Find a counterexample showing that the following conjecture is false: *Let a, b, c, d be natural numbers satisfying $\frac{a}{b} \leq \frac{c}{d}$. Then $a \leq c$ and $b \leq d$.*
18. Find a counterexample showing that the following conjecture is false: *Let $m \geq 0$ and $n \geq 0$ be integers. Then $m + n \leq m \cdot n$.*
19. Find a counterexample showing that the following conjecture is false: *Let $x \geq 0$ and $y \geq 0$ be real numbers. Then $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.*
20. Let a, b, c, d be real numbers. Suppose $a + b = c + d$ and $a \leq c$. Prove that $d \leq b$.
21. Show that $.045000\cdots = .044999\cdots$, where the 0's and the 9's repeat forever.
22. Let a, b, c be real numbers. Prove that $a^2 + b^2 + c^2 \geq ab + bc + ac$.

Exercise Notes: For Exercise 10, if $a \neq b$ then $a - b \neq 0$. For Exercise 20, note that $x \leq y$ if and only if $x - y \leq 0$, for real numbers x and y.

3.2 Using Proof Diagrams as Guides for Proving Theorems

In the previous section we developed techniques for proving theorems about equations and inequalities. For example, in the proof of Theorem 3.1.3 we assumed that $m = 2i + 5$ and $n = 3j$, and then deduced the equation $mn = 6ij + 15j$. Similarly, in our proof of Theorem 3.1.8 we assumed that $a < b$ and $c < d$, and then proved the inequality $a + c < b + d$. These two theorems illustrate the fact that a **mathematical proof** is a logical argument that demonstrates, under certain assumptions, that a

[2]The number φ is called the *golden ratio*. Many artists and architects use the golden ratio φ in their work. When a rectangle of length b and width a satisfies $\frac{b}{a} = \varphi$, it is considered to be more aesthetically pleasing.