Negation Laws for Bounded Number Quantifiers

1. $\neg(\forall x > a)P(x) \Leftrightarrow (\exists x > a)\neg P(x)$.
2. $\neg(\exists x > a)P(x) \Leftrightarrow (\forall x > a)\neg P(x)$.
3. $\neg(\forall x < a)P(x) \Leftrightarrow (\exists x < a)\neg P(x)$.
4. $\neg(\exists x < a)P(x) \Leftrightarrow (\forall x < a)\neg P(x)$.

Similar negation laws apply when the bounded number quantifiers involve the relations $\leq$ and $\geq$. In the above laws, if you move the negation symbol through a bounded number quantifier, then the quantifier changes and the negation symbol completely passes over the relations $x > a$ and $x < a$. For example, we can express $\neg(\forall x > 0)(x^2 > 1 \rightarrow x < 4)$ as a positive statement as follows:

$$
\begin{aligned}
\neg(\forall x > 0)(x^2 > 1 \rightarrow x < 4) &\Leftrightarrow (\exists x > 0)\neg(x^2 > 1 \rightarrow x < 4) && \text{by QNL} \\
&\Leftrightarrow (\exists x > 0)(x^2 > 1 \land x \not< 4) && \text{by CL} \\
&\Leftrightarrow (\exists x > 0)(x^2 > 1 \land x \geq 4) && \text{by laws of inequality.}
\end{aligned}
$$

Exercises 2.3

1. Using quantifier negation laws and propositional logic laws, translate each of the following assertions into positive statements. (The universe is $\mathbb{R}$.)

(a) $\neg(\forall x > 2)(x < 4 \rightarrow x^2 < 16)$.
(b) $\neg(\forall x < 2)(x < 4 \rightarrow x^2 < 16)$.
(c) $\neg(\exists x < 2)(x < 4 \land x^2 < 4)$.
(d) $\neg(\exists x > 2)(x < 4 \land x^2 < 4)$.
(e) $\neg(\forall x \in \mathbb{N})(x > 2 \rightarrow 3x < 2^x)$.
(f) $\neg(\forall x \in \mathbb{N})(x > 4 \rightarrow 3x < 2^x)$.

2. Express the negation of the following statement as a positive statement: *For all real numbers x, if x is rational and positive, then $\sqrt{x}$ is irrational.* State your result in English. [The square root operation $\sqrt{x}$ is defined on page 95.]

3. Consider the following statement and proposed negation of this statement.
Statement: *Every prime number is odd.*
Proposed Negation: *Every prime number is even.*
Determine whether the proposed negation is correct. If it is not correct, then write a correct negation.

4. Using quantifier negation laws and propositional logic laws, express each statement in as a positive statement. (The universe is the set of real numbers.)

(a) $\neg(\forall x > 3)(|x - 10| < \frac{1}{2} \rightarrow |x^2 - 100| < \frac{1}{3})$.
(b) $\neg(\exists x < -4)(|x + 6| < \frac{1}{100} \land |\sin(x) - 100| \geq \frac{1}{30})$.