Definition 2.2.4 (Bounded Number Quantifiers). When our universe is a set of numbers and a is a specific number, we write $(\forall x < a)P(x)$ to mean that *for every number $x < a$, $P(x)$ is true*. Similarly, we write $(\exists x < a)P(x)$ to assert that *for some number $x < a$, $P(x)$ is true*.

Let a be a number. The assertion $(\forall x < a)P(x)$ means that for every number x, if $x < a$ then $P(x)$ is true. Similarly, the statement $(\exists x < a)P(x)$ means that there is a number x such that $x < a$ and $P(x)$ is true. Thus, we have the logical equivalences:

1. $(\forall x < a)P(x) \Leftrightarrow \forall x(x < a \to P(x))$.
2. $(\exists x < a)P(x) \Leftrightarrow \exists x(x < a \wedge P(x))$.

There are similar bounded number quantifiers for the inequalities $\leq$, $>$, $\geq$ as well; for example, the quantifiers in $(\forall x \leq a)P(x)$ and $(\exists x > a)P(x)$ are also referred to as bounded number quantifiers. The statement $(\forall x \leq a)P(x)$ means for every number $x \leq a$, the statement $P(x)$ is true. Similarly, the assertion $(\exists x > a)P(x)$ means that for some number $x > a$ the assertion $P(x)$ is true.

Exercises 2.2

1. Write the statements in logical form, using an appropriate bounded quantifier.

(a) For every real number x, if $x > 1$ then $x > \frac{1}{x}$.
(b) There exists a rational number y such that $y < 2$ and $y^2 > 4$.

2. Determine whether the statements are true or false in the universe $\mathbb{R}$.

(a) $\forall x(x^2 + 1 > 0)$.
(b) $\forall x(x^2 + x \geq 0)$.
(c) $\forall x(x > \frac{1}{2} \to \frac{1}{x} < 3)$.
(d) $\exists x(\frac{1}{x-1} = 3)$.
(e) $\exists x(\frac{1}{x-1} = 0)$.

3. Consider the predicates:

$C(x)$: "x is in the class."
$M(x)$: "x is a mathematics major."

Using these predicates, analyze the logical form of each of the sentences where the universe is the set of all college students.

(a) Everyone in the class is a mathematics major.
(b) Someone in the class is a mathematics major.
(c) No one in the class is a mathematics major.
(d) There a mathematics major who is not in the class.
(e) Every mathematics major is in the class.

4. Let $D = \{-48, -14, -8, -2, 0, 1, 3, 7, 10, 12\}$. Determine which of the following statements are true. If a statement is false, then explain why.

(a) $(\forall x \in D)(\text{if } x \text{ is odd, then } x > 0)$.
(b) $(\forall x \in D)(\text{if } x > 12, \text{ then } x < 0)$.
(c) $(\exists x \in D)(x \text{ is a perfect cube})$. (An integer i is a *perfect cube* if $i = n^3$ for some integer n.)

5. Using the Tarskian predicates in Example 5 on page 38, translate the following English sentences into logical sentences.

(a) Something is white.
(b) Some circle is white.
(c) All squares are black.
(d) No squares are black.
(e) All triangles are west of d.
(f) A triangle is west of d.
(g) There is a triangle that is north of d but not west of a.
(h) Some triangle is not gray.
(i) Every triangle is either west of a or north of b.
(j) No square has the same color as b.

6. Using the Tarskian predicates in Example 5, translate the following five logical sentences into English sentences. Then determine the truth or falsity of each of these statements in the Tarskian world of Fig. 2.1.

(a) $\forall x(I(x) \to (T(x) \vee S(x)))$.
(b) $\forall x(B(x) \to (T(x) \vee S(x)))$.
(c) $\exists y(C(y) \wedge \neg N(y,d))$.
(d) $\exists y(C(y) \wedge N(y,d))$.
(e) $\exists y(C(y) \wedge \neg W(y,g))$.

7. Determine whether the sentences are true or false in the universe $\mathbb{R}$.

(a) $(\forall x > 2)(x < 4 \to x^2 < 16)$.
(b) $(\forall x < 2)(x < 4 \to x^2 < 16)$.
(c) $(\exists x < 2)(x < 4 \wedge x^2 < 4)$.
(d) $(\exists x > 2)(x < 4 \wedge x^2 < 4)$.

8. Determine whether the sentences are true or false.

(a) $(\forall x \in \mathbb{N})(x > 2 \to 3x < 2^x)$.
(b) $(\forall x \in \mathbb{N})(x > 4 \to 3x < 2^x)$.
(c) $(\exists x \in \mathbb{Z})(\frac{1}{5+x} \in \mathbb{N})$.
(d) $(\exists x \in \mathbb{N})(\frac{1}{5+x} \in \mathbb{Z})$.